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Constant-curvature, covariant conservation, and Ricci-scalar evolution in quadratic Ricci gravity and FLRW cosmology

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Abstract
Higher-curvature extensions of general relativity provide a mathematically natural framework for investigating gravitational dynamics beyond the Einstein–Hilbert theory. This paper develops three complementary results concerning the Ricci tensor and Ricci scalar in four-dimensional spacetime. First, we establish a constant-curvature theorem for the quadratic Ricci theory defined by the action $R + \alpha R_{\mu\nu}R_{\mu\nu} - 2\Lambda$.
We prove that, on an exactly four-dimensional maximally symmetric vacuum background, the Euler–Lagrange tensor generated by the quadratic Ricci invariant vanishes identically. Consequently, the background field equation reduces to the Einstein equation and the sectional curvature is $K = \Lambda/3$, independently of the finite value of the quadratic coupling α. Second, we establish the covariant conservation of the quadratic Ricci Euler–Lagrange tensor. This result follows from the contracted Bianchi identity together with the diffeomorphism invariance of the quadratic-curvature action and guarantees the differential consistency of the extended gravitational equations. Third, we derive the evolution law for the Ricci scalar in a four-dimensional Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime containing a homogeneous and isotropic perfect fluid. For a barotropic equation of state $p = w\rho$, we show that

$$\dot{R} = -24\pi GH(1 - 3w)(1 + w)\rho,$$

and identify the conditions under which the Ricci scalar is dynamically constant. In particular, radiation $(w = 1/3)$ has no matter contribution to R, whereas an expanding dust universe $(w = 0, H > 0, \rho > 0)$ has a strictly decreasing Ricci scalar. Together, the results demonstrate that quadratic Ricci corrections can leave an exact maximally symmetric background unchanged while remaining dynamically relevant for perturbations and non-maximally symmetric geometries. The results provide a unified mathematical connection between higher-curvature gravitational theories, geometric conservation laws, and cosmological curvature evolution.

Keywords: Ricci tensor, Ricci scalar, quadratic gravity, higher-curvature gravity, maximally symmetric spacetime, covariant conservation, Bianchi identity, FLRW cosmology, cosmological constant, perfect fluid

1. Introduction
General relativity describes gravitation geometrically by relating spacetime curvature to the distribution and flow of matter. Its central dynamical equation is the Einstein field equation,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the energy-momentum tensor, G is Newton's gravitational constant, and Λ is the cosmological constant. The geometric consistency of this equation follows from the contracted

Bianchi identity, $\nabla^{\mu} G_{\mu\nu} = 0$, which implies covariant conservation of the energy-momentum tensor in the minimally coupled theory (Wald, 1984) [7].

Although general relativity has been highly successful, higher-curvature extensions of gravity have received considerable attention in both classical and quantum gravitational research. In four dimensions, terms quadratic in curvature arise naturally in effective gravitational actions and play an important role in studies of renormalization, ultraviolet behavior, cosmological dynamics, and modified gravitational propagation (Stelle, 1977, 1978) [5, 6]. General reviews of modified gravity emphasize that higher-order curvature terms can produce additional gravitational degrees of freedom and modify the dynamics away from the Einsteinian regime (Clifton et al., 2012) [1].

One particularly simple higher-curvature model is obtained by adding the Ricci invariant $R_{\mu\nu}R^{\mu\nu}$ to the Einstein–Hilbert action. The corresponding action considered in this paper is

$$S = \frac{1}{16\pi G}\int_M (R + \alpha R_{\mu\nu}R^{\mu\nu} - 2\Lambda)\sqrt{-g}\, d^4x, \tag{1}$$

where α is a constant coupling parameter. The inclusion of the quadratic term produces higher-order differential contributions to the metric field equations. Such theories constitute an important class of quadratic-curvature gravitational models (Stelle, 1977, 1978; Lü & Pope, 2011) [2, 5, 6].

The mathematical structure of these theories raises several important questions. First, does the quadratic Ricci correction necessarily change the background geometry of a highly symmetric vacuum spacetime? Second, is the additional curvature tensor compatible with the differential conservation laws underlying general relativity? Third, how does the Ricci scalar evolve in a cosmological spacetime filled with a perfect fluid?

These questions are closely connected. Maximally symmetric spaces provide the simplest exact backgrounds on which higher-curvature field equations can be evaluated. Their curvature tensor is completely determined by a single constant sectional curvature. This strong geometric restriction causes many derivative terms to disappear and allows the algebraic structure of the quadratic correction to be examined explicitly. Similar constant-curvature backgrounds are central to analyses of higher-derivative and critical gravity (Lü & Pope, 2011; Pravda et al., 2017) [2, 4].

The second issue concerns the differential consistency of the modified field equations. Since the Einstein tensor satisfies the contracted Bianchi identity, any geometrically consistent extension of Einstein gravity must possess a corresponding conservation structure. In a diffeomorphism-invariant theory, this structure is inherited from the invariance of the action. Thus, the quadratic Ricci Euler–Lagrange tensor $Q_{\mu\nu}$, is not an arbitrary tensorial correction; it satisfies a differential identity ensuring compatibility with the gravitational equations.

The third issue concerns cosmology. In an FLRW universe, homogeneity and isotropy reduce the gravitational dynamics to functions of cosmic time. The Ricci scalar therefore becomes a useful scalar diagnostic of the evolving geometry. Taking the trace of Einstein's equation relates R directly to the trace of the matter energy-momentum tensor. For a perfect fluid this produces a particularly simple relationship between curvature, density, pressure, and the cosmological constant (Mukhanov, 2005) [3].

The purpose of this paper is therefore to establish and interpret three results:

1. a constant-curvature theorem for four-dimensional quadratic Ricci gravity;
2. a covariant conservation theorem for the quadratic Ricci correction;
3. a Ricci-scalar evolution theorem for FLRW cosmology.

The first theorem demonstrates that the quadratic Ricci correction vanishes on an exact maximally symmetric vacuum background. The second establishes the differential consistency of the extended field equations. The third connects the Ricci scalar to the cosmological equation of state and identifies conditions for constant and monotonic curvature evolution.

2. Preliminaries

2.1 Lorentzian spacetime

Let (M, g) be a smooth four-dimensional Lorentzian manifold with metric signature

$$(-,+,+,+). \tag{2}$$

The Levi–Civita connection associated with $g_{\mu\nu}$ is denoted by ∇_{μ}. It is torsion-free and metric-compatible:

$$\nabla_{\lambda} g_{\mu\nu} = 0. \tag{3}$$

The Riemann curvature tensor is defined by the convention

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{4}$$

The Ricci tensor is obtained by contraction,

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}, \tag{5}$$

and the Ricci scalar is

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{6}$$

The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \quad (7)$$

2.2 Differential Bianchi identity

The Riemann tensor satisfies the differential Bianchi identity

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\rho R_{\mu\nu\sigma\lambda} + \nabla_\sigma R_{\mu\nu\lambda\rho} = 0. \quad (8)$$

Contracting appropriate pairs of indices yields

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R. \quad (9)$$

Consequently,

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla^\mu\left(Rg_{\mu\nu}\right). \quad (10)$$

Using metric compatibility,

$$\nabla^\mu\left(Rg_{\mu\nu}\right) = g_{\mu\nu}\nabla^\mu R = \nabla_\nu R, \quad (11)$$

and hence

$$\nabla^\mu G_{\mu\nu} = 0. \quad (12)$$

Equation (12) is the contracted Bianchi identity and is fundamental to the consistency of general relativity (Wald, 1984) [7].

2.3 Quadratic Ricci gravity

We consider the action

$$S = \frac{1}{16\pi G}\int_M (R + \alpha R_{\mu\nu}R^{\mu\nu} - 2\Lambda)\sqrt{-g}\, d^4x. \quad (13)$$

The corresponding vacuum field equation is written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha Q_{\mu\nu} = 0, \quad (14)$$

where $Q_{\mu\nu}$ denotes the Euler–Lagrange tensor generated by the quadratic Ricci invariant.
For the curvature convention adopted in this paper, we use

$$Q_{\mu\nu} = 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \nabla_\mu\nabla_\nu R + \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(\Box R - R_{\alpha\beta}R^{\alpha\beta}), \quad (15)$$

where

$$\Box = \nabla^\alpha\nabla_\alpha \quad (16)$$

is the covariant d'Alembert operator.
The precise signs of curvature-dependent expressions depend on the Riemann-tensor convention. Equation (15) therefore fixes the convention used throughout the remainder of this paper.

2.4 Maximally symmetric spacetime

A four-dimensional maximally symmetric spacetime has constant sectional curvature K. Its Riemann tensor is

$$R_{\mu\alpha\nu\beta} = K\left(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu}\right). \quad (17)$$

Contracting α and β,

$$R_{\mu\nu} = 3Kg_{\mu\nu}. \quad (18)$$

Taking another trace gives

$$R = g^{\mu\nu}R_{\mu\nu} = 12K. \quad (19)$$

Thus,

$$R_{\mu\nu} = 3Kg_{\mu\nu}, R = 12K. \quad (20)$$

These relations characterize the curvature of a four-dimensional maximally symmetric spacetime.

3. Methodology and Mathematical Framework

The analysis is entirely analytical. The three theorems are established by tensor contraction, covariant differentiation, curvature identities, and the cosmological conservation law.

The methodology consists of four principal stages.

First, the quadratic Ricci action is varied with respect to the metric to obtain the modified vacuum field equations. The resulting Euler–Lagrange tensor $Q_{\mu\nu}$ is then evaluated on a maximally symmetric background.

Second, the differential Bianchi identity and diffeomorphism invariance are used to establish the covariant conservation of $Q_{\mu\nu}$.

Third, the trace of Einstein's equations is evaluated for a perfect fluid in an FLRW spacetime. This produces an algebraic expression for the Ricci scalar in terms of $\rho, p,$ and Λ .

Fourth, the cosmological conservation equation is used to determine the time derivative of the Ricci scalar and its sign for different equations of state.

The analysis assumes smooth fields and a Levi–Civita connection throughout.

4. Main Results

4.1 Theorem 1: Constant-Ricci-Curvature Theorem in Quadratic Ricci Gravity

Theorem 1

Let (M, g) be a four-dimensional Lorentzian spacetime governed in vacuum by the quadratic Ricci action

$$S = \frac{1}{16\pi G}\int_M (R + \alpha R_{\mu\nu}R^{\mu\nu} - 2\Lambda)\sqrt{-g}\, d^4x. \quad (21)$$

Suppose that (M, g) is maximally symmetric with constant sectional curvature K. Then

$$Q_{\mu\nu} = 0. \quad (22)$$

Consequently, the vacuum field equation reduces to

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \quad (23)$$

and the sectional curvature is

$$K = \frac{\Lambda}{3}. \quad (24)$$

Hence,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, R = 4\Lambda. \quad (25)$$

Thus, the quadratic coupling α does not alter the exact maximally symmetric vacuum background.

Proof

The vacuum field equation derived from (21) is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha Q\mu\nu = 0. \quad (26)$$

For a maximally symmetric four-dimensional spacetime,

$$R_{\mu\alpha\nu\beta} = K\left(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu}\right). \quad (27)$$

Contracting the first and third curvature indices gives

$$R_{\mu\nu} = 3Kg_{\mu\nu}. \quad (28)$$

Taking the trace,

$$R = g^{\mu\nu}R_{\mu\nu} = 3Kg^{\mu\nu}g_{\mu\nu}. \quad (29)$$

Since the spacetime dimension is four,

$$g^{\mu\nu}g_{\mu\nu} = 4, \tag{30}$$

so

$$R = 12K. \tag{31}$$

Because K is constant and the connection is metric-compatible,

$$\nabla_\lambda R_{\mu\nu} = \nabla_\lambda\left(3Kg_{\mu\nu}\right) = 3(\nabla_\lambda K)g_{\mu\nu} + 3K\nabla_\lambda g_{\mu\nu} = 0. \tag{32}$$

Similarly,

$$\nabla_\lambda R = \nabla_\lambda(12K) = 12\nabla_\lambda K = 0. \tag{33}$$

Therefore,

$$\nabla_\nu R = 0, \tag{34}$$

and

$$\nabla_\mu\nabla_\nu R = 0. \tag{35}$$

Also,

$$\nabla_\mu\nabla_\nu R_{\mu\nu} = 0. \tag{36}$$

It remains to evaluate the algebraic terms in $Q_{\mu\nu}$.
From (28),

$$R_{\alpha\beta} = 3Kg_{\alpha\beta}. \tag{37}$$

Hence,

$$R_{\mu\alpha\nu\beta}R^{\alpha\beta} = 3KR_{\mu\alpha\nu\beta}g^{\alpha\beta}. (\tag{38}$$

Using (28),

$$R_{\mu\alpha\nu\beta}g^{\alpha\beta} = R_{\mu\nu} = 3Kg_{\mu\nu}. \tag{39}$$

Consequently,

$$R_{\mu\alpha\nu\beta}R^{\alpha\beta} = 9K^2g_{\mu\nu}. \tag{40}$$

Therefore,

$$2R_{\mu\alpha\nu\beta}R^{\alpha\beta} = 18K^2g_{\mu\nu}. \tag{41}$$

Next,

$$R_{\alpha\beta}R^{\alpha\beta} = \left(3Kg_{\alpha\beta}\right)\left(3Kg^{\alpha\beta}\right). \tag{42}$$

Using (30),

$$R_{\alpha\beta}R^{\alpha\beta} = 9K^2(4) = 36K^2. \tag{43}$$

The derivative terms in (15) vanish by (34)–(36). Therefore, (15)

$$Q_{\mu\nu} = 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \nabla_\mu\nabla_\nu R + \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(\Box R - R_{\alpha\beta}R^{\alpha\beta}),$$

Reduces to

$$Q_{\mu\nu} = 18K^2g_{\mu\nu} - 0 + 0 + \tfrac{1}{2}g_{\mu\nu}(0 - 36K^2). \tag{44}$$

Thus,

$$Q_{\mu\nu} = 18K^2 g_{\mu\nu} - 18K^2 g_{\mu\nu} = 0. \quad (45)$$

Hence,

$$Q_{\mu\nu} = 0. \quad (46)$$

Substituting (46) into the modified field equation (26),

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \quad (47)$$

Using

$$R_{\mu\nu} = 3K g_{\mu\nu}, R = 12K, \quad (48)$$

The Einstein tensor becomes

$$G_{\mu\nu} = 3K g_{\mu\nu} - \frac{1}{2}(12K) g_{\mu\nu}. \quad (49)$$

Therefore,

$$G_{\mu\nu} = -3K g_{\mu\nu}. \quad (50)$$

Equation (47) becomes

$$(-3K + \Lambda) g_{\mu\nu} = 0. \quad (51)$$

Since $g_{\mu\nu}$ is non-degenerate,

$$-3K + \Lambda = 0. \quad (52)$$

Thus,

$$K = \frac{\Lambda}{3}. \quad (53)$$

Substituting this into (48),

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \quad (54)$$

and

$$R = 4\Lambda. \quad (55)$$

This completes the proof.

4.2 Corollary 1: Independence of the Exact Background from α

Corollary

Under the assumptions of Theorem 1, the quadratic coupling α does not modify the exact maximally symmetric vacuum curvature.

Proof

Theorem 1 establishes

$$Q_{\mu\nu} = 0. \quad (56)$$

Therefore,

$$\alpha Q_{\mu\nu} = 0 \quad (57)$$

for every finite value of α. The field equation consequently becomes

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \quad (58)$$

Thus,

$$K = \frac{\Lambda}{3}, R = 4\Lambda. \tag{59}$$

The result does not imply that quadratic gravity is identical to general relativity. It establishes only that the particular exact maximally symmetric vacuum background is unchanged. This distinction is important because higher-curvature corrections generally contribute to perturbations and to geometries that are not maximally symmetric (Clifton et al., 2012; Lü & Pope, 2011). [1, 2]

5. Covariant Conservation of the Quadratic Ricci Correction

5.1 Theorem 2: Divergence-Free Property

Theorem 2

Let $Q_{\mu\nu} = 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \nabla_\mu\nabla_\nu R + \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(\Box R - R_{\alpha\beta}R^{\alpha\beta})$ (60)

be the Euler–Lagrange tensor associated with the Ricci-squared invariant in the four-dimensional action (21). Then

$$\nabla^\mu Q_{\mu\nu} = 0. \tag{61}$$

Consequently, the complete geometric tensor

$$\mathcal{H}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha Q_{\mu\nu} \tag{62}$$

satisfies

$$\nabla^\mu \mathcal{H}_{\mu\nu} = 0. \tag{63}$$

Proof

Starting with the vacuum field equation derived from diffeomorphism invariance of the action (21) we have:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha Q_{\mu\nu} = 0. \tag{64}$$

Take the covariant divergence:

$$\nabla^\mu\left(G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha Q_{\mu\nu}\right) = 0. \tag{65}$$

By linearity,

$$\nabla^\mu G_{\mu\nu} + \Lambda\nabla^\mu g_{\mu\nu} + \alpha\nabla^\mu Q_{\mu\nu} = 0. \tag{66}$$

The contracted Bianchi identity gives

$$\nabla^\mu G_{\mu\nu} = 0, \tag{67}$$

while metric compatibility gives

$$\nabla^\mu g_{\mu\nu} = 0. \tag{68}$$

Therefore,

$$\alpha\nabla^\mu Q_{\mu\nu} = 0. \tag{69}$$

For nonzero α,

$$\nabla^\mu Q_{\mu\nu} = 0. \tag{70}$$

Hence,

$$\nabla^\mu Q_{\mu\nu} \equiv 0. \tag{71}$$

Consequently,

$$\nabla^\mu\left(G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha Q_{\mu\nu}\right) = 0. \tag{72}$$

This proves the theorem.

5.2 Non-vacuum extension

The corresponding field equation with matter is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha Q_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{73}$$

Taking the covariant divergence,

$$\nabla^{\mu} G_{\mu\nu} + \Lambda \nabla^{\mu} g_{\mu\nu} + \alpha \nabla^{\mu} Q_{\mu\nu} = 8\pi G \nabla^{\mu} T_{\mu\nu}. \tag{74}$$

Using,

$$\nabla^{\mu} G_{\mu\nu} = 0, \tag{75}$$

$$\nabla^{\mu} g_{\mu\nu} = 0. \tag{76}$$

And

$$\nabla^{\mu} Q_{\mu\nu} = 0, \tag{77}$$

we obtain the conservation law

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{78}$$

Thus, the quadratic Ricci correction does not introduce an independent violation of energy-momentum conservation when the matter sector is minimally coupled.
This conservation structure is one of the essential requirements for a consistent geometric extension of general relativity (Clifton et al., 2012) [1].

6. Ricci-Scalar Evolution in FLRW Cosmology

6.1 FLRW Geometry

Consider the four-dimensional FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \tag{79}$$

where $a(t)$ is the scale factor, $k = 0, \pm 1$ is the normalized spatial curvature parameter, and

$$d\Omega^2 = d\theta^2 + \sin^2\theta \; d\phi^2. \tag{80}$$

The Hubble parameter is

$$H = \frac{\dot{a}}{a}. \tag{81}$$

The homogeneous and isotropic matter source is represented by the perfect-fluid energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + p g_{\mu\nu}, \tag{82}$$

with

$$u^{\mu}u_{\mu} = -1. \tag{83}$$

The trace is

$$T = g^{\mu\nu} T_{\mu\nu}. \tag{84}$$

7. Theorem 3: Ricci-Tensor Trace Evolution Theorem

Theorem 3

Let (M, g) be a four-dimensional FLRW spacetime with signature $(-+++)$, scale factor $a(t)$, Hubble parameter $H = \dot{a}/a$, and homogeneous and isotropic perfect-fluid matter satisfying

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + p g_{\mu\nu}. \tag{85}$$

Suppose Einstein's field equations with cosmological constant are satisfied:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (86)$$

Then the Ricci scalar is

$$R = 4\Lambda + 8\pi G(\rho - 3p). \quad (87)$$

Its cosmic-time evolution is

$$\dot{R} = 8\pi G(\dot{\rho} - 3\dot{p}). \quad (88)$$

If the fluid obeys the barotropic equation of state

$$p = w\rho, \quad (89)$$

where w is constant, then

$$\dot{R} = -24\pi GH(1 - 3w)(1 + w)\rho. \quad (90)$$

Consequently, R is dynamically constant whenever

$$w = \frac{1}{3} \; or \; w = -1, \quad (91)$$

apart from the trivial cases $H = 0 \; or \; \rho = 0$.

Proof

Starting with Einstein's field equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (92)$$

contract with $g^{\mu\nu}$:

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} + \Lambda g^{\mu\nu}g_{\mu\nu} = 8\pi G g^{\mu\nu}T_{\mu\nu}. \quad (93)$$

By definition,

$$g^{\mu\nu}R_{\mu\nu} = R. \quad (94)$$

Since the spacetime dimension is four,

$$g^{\mu\nu}g_{\mu\nu} = 4. \quad (95)$$

Therefore,

$$R - 2R + 4\Lambda = 8\pi GT. \quad (96)$$

Hence,

$$-R + 4\Lambda = 8\pi GT, \quad (97)$$

and therefore

$$R = 4\Lambda - 8\pi GT. \quad (98)$$

For the perfect fluid (85),

$$T = g^{\mu\nu}\left[(\rho + p)u_\mu u_\nu + pg_{\mu\nu}\right]. \quad (99)$$

Thus,

$$T = (\rho + p)u^\mu u_\mu + pg^{\mu\nu}g_{\mu\nu}. \quad (100)$$

Using

$$u^{\mu}u_{\mu} = -1 \tag{101}$$

and

$$g^{\mu\nu}g_{\mu\nu} = 4, \tag{102}$$

we obtain

$$T = -(\rho + p) + 4p. \tag{103}$$

Therefore,

$$T = -\rho + 3p. \tag{104}$$

Substitution into (98) gives

$$R = 4\Lambda - 8\pi G(-\rho + 3p). \tag{105}$$

Consequently,

$$R = 4\Lambda + 8\pi G(\rho - 3p). \tag{106}$$

This establishes the first result.
Because G and Λ are constants, differentiating equation (106) gives

$$\dot{R} = 8\pi G(\dot{\rho} - 3\dot{p}). \tag{107}$$

Now impose the barotropic equation of state

$$p = w\rho, \tag{108}$$

with constant w. Differentiating,

$$\dot{p} = w\dot{\rho}. \tag{109}$$

Hence,

$$\dot{R} = 8\pi G(1 - 3w)\dot{\rho}. \tag{110}$$

The conservation law

$$\nabla^{\mu}T_{\mu\nu} = 0 \tag{111}$$

for a homogeneous and isotropic FLRW perfect fluid gives the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{112}$$

Therefore,

$$\dot{\rho} = -3H(\rho + p). \tag{113}$$

Using $p = w\rho$,

$$\dot{\rho} = -3H(1 + w)\rho. \tag{114}$$

Substituting (114) into (110),

$$\dot{R} = 8\pi G(1 - 3w)[-3H(1 + w)\rho]. \tag{115}$$

Therefore,

$$\dot{R} = -24\pi GH(1 - 3w)(1 + w)\rho. \tag{116}$$

For a nontrivial cosmological evolution with $H\rho \neq 0$, the Ricci scalar is constant when

$$(1-3w)(1+w)=0. \tag{117}$$

Hence,

$$w=\frac{1}{3} \tag{118}$$

or

$$w=-1. \tag{119}$$

This proves the theorem.

8. Corollaries of the FLRW Ricci-Scalar Theorem

8.1 Corollary 1: Radiation

For radiation,

$$w=\frac{1}{3}. \tag{120}$$

Equation (87) gives

$$R=4\Lambda+8\pi G\left(\rho-\frac{3\rho}{3}\right). \tag{121}$$

Therefore,

$$R=4\Lambda. \tag{122}$$

In particular, when

$$\Lambda=0, \tag{123}$$

we obtain

$$R=0. \tag{124}$$

Thus, although radiation possesses nonzero energy density, its energy-momentum tensor is traceless:

$$T=0, \tag{125}$$

so it makes no direct contribution to the Ricci scalar.

8.2 Corollary 2: Cosmological-Constant Fluid

For

$$w=-1, \tag{126}$$

we have

$$p=-\rho. \tag{127}$$

Equation (114) becomes

$$\dot{\rho}=-3H(1-1)\rho=0. \tag{128}$$

Thus the density is constant. Equation (87) gives

$$R=4\Lambda+8\pi G[\rho-3(-\rho)]. \tag{129}$$

Hence,

$$R=4\Lambda+32\pi G\rho, \tag{130}$$

which is constant.

This corresponds to the geometric behavior expected from a vacuum-energy-dominated spacetime.

8.3 Corollary 3: Dust-Dominated FLRW Universe
For pressureless matter,

$$p = 0, \tag{131}$$

so

$$w = 0. \tag{132}$$

Equation (87) becomes

$$R = 4\Lambda + 8\pi G\rho. \tag{133}$$

The continuity equation becomes

$$\dot{\rho} + 3H\rho = 0, \tag{134}$$

or

$$\dot{\rho} = -3H\rho. \tag{135}$$

Differentiating (133),

$$\dot{R} = 8\pi G\dot{\rho}. \tag{136}$$

Therefore,

$$\dot{R} = -24\pi GH\rho. \tag{137}$$

For an expanding universe,

$$H > 0, \tag{138}$$

and positive matter density,

$$\rho > 0, \tag{139}$$

we obtain

$$\dot{R} < 0. \tag{140}$$

Thus, the Ricci scalar is strictly decreasing during an expanding positive-density dust-dominated epoch.

9. Physical and Geometrical Interpretation
The three theorems reveal complementary properties of curvature in higher-order gravitational and cosmological settings.

9.1 Exact maximally symmetric backgrounds
Theorem 1 shows that the quadratic Ricci contribution vanishes on a four-dimensional maximally symmetric vacuum background:

$$Q_{\mu\nu} = 0. \tag{141}$$

This is a nontrivial cancellation. The action contains the higher-order invariant

$$R_{\mu\nu}R^{\mu\nu}, \tag{142}$$

yet its metric variation contributes no net tensor on the exact constant-curvature background.
Consequently,

$$K = \frac{\Lambda}{3}, \tag{143}$$

and

$$R = 4\Lambda. \tag{144}$$

The result is therefore an example of background degeneracy: the higher-curvature theory and Einstein gravity possess the same exact maximally symmetric vacuum geometry.
This should not be interpreted as saying that the two theories have identical physics. Higher-curvature terms generally become important when the spacetime departs from maximal symmetry. Perturbations about an Einstein background can activate additional higher-derivative contributions and, depending on the complete quadratic theory, additional propagating modes (Stelle, 1977, 1978; Lü & Pope, 2011) [2, 5, 6].

9.2 Differential consistency
Theorem 2 establishes

$$\nabla^{\mu} Q_{\mu\nu} = 0. \tag{145}$$

This means that the quadratic correction is itself geometrically compatible with covariant conservation.
The modified field equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha Q_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{146}$$

therefore satisfies

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{147}$$

The result is particularly important because higher-order gravitational equations contain derivatives of curvature, and one might otherwise expect the additional terms to introduce an independent consistency condition. Instead, diffeomorphism invariance ensures the required differential identity.

9.3 Cosmological curvature evolution
Theorem 3 shows that the Ricci scalar is controlled by the trace of the matter energy-momentum tensor:

$$R = 4\Lambda + 8\pi G(\rho - 3p). \tag{148}$$

This reveals an important distinction between different cosmological fluids.

For dust,

$$p = 0,$$

So matter contributes directly to R.
For radiation,

$$p = \frac{\rho}{3},$$

and therefore

$$\rho - 3p = 0.$$

Hence radiation does not contribute directly to the Ricci scalar.

For a cosmological-constant fluid,

$$p = -\rho,$$

and the trace is nonzero and constant.
The evolution equation

$$\dot{R} = -24\pi G H(1 - 3w)(1 + w)\rho \tag{149}$$

therefore provides a compact curvature diagnostic for barotropic cosmological matter.

10. Relation Between the Three Results
The three theorems can be viewed as describing different levels of geometric structure.
The first theorem is an algebraic background theorem. It establishes what happens when the curvature is maximally constrained:

$R_{\mu\nu} = 3K g_{\mu\nu}$.

The quadratic correction then cancels exactly.
The second theorem is a differential consistency theorem. It establishes that

$\nabla^{\mu} Q_{\mu\nu} = 0,$

ensuring that the additional curvature contribution is compatible with the geometric conservation structure of the theory.
The third theorem is a dynamical cosmological theorem. It describes how the scalar curvature changes when matter evolves:

$\dot{R} = -24\pi GH(1 - 3w)(1 + w)\rho.$

Together they establish the following hierarchy:

Maximal symmetry $\Longrightarrow Q_{\mu\nu} = 0,$ (150)

Diffeomorphism invariance $\Longrightarrow \nabla^{\mu} Q_{\mu\nu} = 0,$ (151)

and

FLRW matter evolution $\Longrightarrow \dot{R} = -24\pi GH(1 - 3w)(1 + w)\rho.$ (152)

These results demonstrate that higher-curvature gravity should not be viewed simply as an arbitrary modification of Einstein's equation. Its geometric structure is strongly constrained by symmetry, differential identities, and the variational origin of the field equations.

11. Discussion

The results obtained here have several implications for mathematical gravitational theory.
First, the vanishing of $Q_{\mu\nu}$ on maximally symmetric backgrounds provides a useful exact solution-generating principle. If

$K = \frac{\Lambda}{3},$

then the background is simultaneously a solution of the quadratic Ricci theory and the Einstein theory with cosmological constant. This property is useful when studying perturbations because one can linearize the higher-curvature equations about a known Einstein background. Higher-derivative corrections may then become visible only at the level of perturbations rather than the zeroth-order geometry.
This observation is consistent with the broader literature on quadratic and critical gravity, where Einstein backgrounds play an important role in determining the spectrum and stability of higher-curvature theories (Lü & Pope, 2011; Pravda et al., 2017) [2, 4]. Recent work continues to investigate perturbations and exact solutions in quadratic gravity, demonstrating that higher-curvature corrections can have substantial effects away from the maximally symmetric limit.
Second, the conservation theorem provides a geometric consistency test for the quadratic theory. Since

$\nabla^{\mu} Q_{\mu\nu} = 0,$

the additional gravitational contribution can be incorporated into an effective geometric tensor,

$G_{\mu\nu}^{\text{eff}} = G_{\mu\nu} + \alpha Q_{\mu\nu}.$ (153)

The field equation can then be expressed as

$G_{\mu\nu}^{\text{eff}} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$ (154)

The effective geometric tensor obeys

$\nabla^{\mu} G_{\mu\nu}^{\text{eff}} = 0.$ (155)

Consequently, the matter sector remains covariantly conserved provided the coupling to matter is minimal.
Third, the FLRW theorem shows that scalar curvature contains direct information about the matter equation of state. Since

$R - 4\Lambda = 8\pi G(\rho - 3p),$ (156)

the quantity $R - 4\Lambda$ measures the trace component of the matter source.
For a barotropic fluid,

$$R - 4\Lambda = 8\pi G(1 - 3w)\rho. \quad (157)$$

Therefore, the sign and magnitude of the curvature contribution depend on the factor $(1 - 3w)$. Radiation is special because $w = 1/3$, producing a traceless energy-momentum tensor.
For dust,

$$R - 4\Lambda = 8\pi G\rho, \quad (158)$$

and the expansion of the universe causes ρ and R to decrease.
These results also illustrate why background calculations alone cannot fully characterize higher-curvature theories. A theory may reproduce the same exact de Sitter or anti-de Sitter background as general relativity while producing different perturbation equations, different gravitational-wave propagation, or different stability properties. Higher-derivative gravitational theories are therefore best studied at both background and perturbative levels (Clifton et al., 2012; Stelle, 1977, 1978) [1, 5, 6].

12. Limitations and Scope

The present analysis is analytical and is restricted to four-dimensional metric gravity with a Levi–Civita connection.
The constant-curvature theorem assumes exact maximal symmetry. It therefore does not establish the vanishing of $Q_{\mu\nu}$ for arbitrary Einstein spaces or general curved backgrounds.
Similarly, the cosmological theorem is derived within standard Einstein gravity rather than the full quadratic Ricci field equations. It therefore describes the FLRW Ricci-scalar evolution of the Einstein sector. A full treatment of FLRW dynamics directly within quadratic Ricci gravity would require substitution of the FLRW metric into the complete higher-derivative field equations.
The conclusions also depend on the curvature-sign convention specified in Equation (4). Changing the Riemann-tensor convention changes the signs of some intermediate curvature expressions, although the physical conclusions can be reformulated consistently.
Finally, the conservation theorem assumes a diffeomorphism-invariant gravitational action and minimally coupled matter. Nonminimal matter-curvature couplings can modify the interpretation of matter conservation.

13. Conclusions

This paper has established three complementary results concerning Ricci curvature in quadratic gravity and cosmology.
The first result is the constant-curvature theorem. For the four-dimensional quadratic Ricci action

$$S = \frac{1}{16\pi G}\int (R + \alpha R_{\mu\nu}R^{\mu\nu} - 2\Lambda)\sqrt{-g}\, d^4x,$$

the quadratic Euler–Lagrange tensor vanishes identically on an exact maximally symmetric vacuum background:

$$Q_{\mu\nu} = 0.$$

Consequently,

$$K = \frac{\Lambda}{3}, R_{\mu\nu} = \Lambda g_{\mu\nu}, R = 4\Lambda.$$

Thus, the quadratic Ricci coupling does not modify the exact maximally symmetric vacuum geometry.
The second result is the covariant conservation theorem:

$$\nabla^{\mu}Q_{\mu\nu} = 0.$$

This establishes that the quadratic Ricci correction is geometrically compatible with the contracted Bianchi identity. Consequently, the complete modified gravitational tensor is covariantly conserved, and minimally coupled matter continues to satisfy

$$\nabla^{\mu}T_{\mu\nu} = 0.$$

The third result is the Ricci-scalar evolution theorem for FLRW cosmology. For a perfect fluid,

$$R = 4\Lambda + 8\pi G(\rho - 3p).$$

For a constant barotropic parameter w,

$$\dot{R} = -24\pi GH(1 - 3w)(1 + w)\rho.$$

This establishes that R is dynamically constant for $w = 1/3$ and $w = -1$, apart from trivial cases. For an expanding dust universe with $H > 0$ and $\rho > 0$,

$\dot{R} < 0,$

so the Ricci scalar decreases monotonically.

The combined significance of these results is that they distinguish three important aspects of curvature dynamics: background geometry, differential consistency, and cosmological evolution. Quadratic curvature corrections may disappear on a highly symmetric exact background while remaining important for perturbations and less symmetric geometries. At the same time, the variational structure of the theory guarantees covariant conservation, while the trace of Einstein's equation provides a direct connection between matter content and scalar curvature in cosmology.

The results therefore provide a rigorous mathematical foundation for further investigations of perturbations, stability, gravitational-wave propagation, and cosmological solutions in quadratic Ricci gravity.

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