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Modeling the Role of Fossil Fuels and Human Activities in Carbon (IV) Oxide Emissions and Global Warming

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Modeling the Role of Fossil Fuels and Human Activities in Carbon (IV) Oxide Emissions and Global Warming

Abstract

This study addresses the critical issue of Carbon (IV) Oxide (CO_2) emissions, a major contributor to global warming and climate change, and aligns with Sustainable Development Goal (SDG) 13 (Climate Action). While many previous studies focus on CO_2 emissions at regional or national levels, this research introduces a global compartmental model to analyze interactions between human activity and fossil fuel consumption. Using a system of ordinary differential equations solved through the Runge-Kutta 45 method and simulated in MATLAB, the study investigates how various factors contribute to CO_2 emissions. Results show that increase in human population and increased fossil fuel use leads to rise in CO_2 emission. Data fitting based on Newton's Law of Cooling suggests that global temperature could rise by up to 6.57°C, though more realistic estimates range between 0.59°C and 2.46°C over two years. Importantly, increasing plant cover could boost CO_2 absorption, offering a significant mitigation strategy. The findings support global efforts to reduce emissions and combat climate-related threats such as rising sea levels, extreme weather, and health risks.

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Keywords: Emissions, Carbon (iv) oxide, Modelling, Simulations, Global warming

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1 Introduction

Greenhouse gas emissions refer to the release of gases that contribute to the greenhouse effect and global warming. The primary greenhouse gases include carbon dioxide (CO_2), methane (CH_4), nitrous oxide (N_2O), and fluorinated gases. These gases are mainly produced by activities such as burning fossil fuels, deforestation, and various industrial processes (Munsif and Zubair 2021). The release of these gases is relatively small compared to CO_2. According to the United States Environmental Protection Agency's 2020 overview, the combustion of fossil fuels like gasoline and diesel was the primary source of emissions, accounting for approximately 33% of total U.S. emissions (Albuquerque et al. 2020)..

The Industrial Revolution in the 18th century led to widespread extraction and use of fossil fuels, essential for transportation, heating, industry, and electricity, but also contributes to climate change (Kundzewicz 2014; Donald et al. 2024). Human activity is causing global warming, as evidenced by increasing greenhouse gases, positive radiative forcing, observed warming, and our growing understanding of the climate system (IPCC 2007).

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Earth warms due to emissions that act like a blanket in the atmosphere, trapping heat. This layer hinders the Earth's ability to cool down, resulting in rising global temperatures (Sasi et al. 2024). Such changes can significantly affect weather patterns, climate conditions, and the availability of food and water resources (Klufallah et al. 2014). Mathematical models represent real-world phenomena using equations and abstractions. Mathematical models represent real-world phenomena using equations and abstractions. They often require simplifications to address complexities. The application of mathematical models to illustrate CO_2 emissions and environmental effects is a recent development, influenced by various scientists and organizations, and difficult to trace (Yoro and Daramola 2020).

A framework known as a compartmental model views a system as a collection of compartments. It attempts to comprehend the behavior by examining the movement of

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elements between these compartments. Every item is in one of the model's specified compartments (Mandal et al. 2021).

Nkuturum et.al. (2022) investigated the dynamics of climate change resulting from CO_2 emissions and assessed the effectiveness of control measures using a compartmental model with ordinary differential equations. The findings indicated that environmental levies and awareness programs effectively reduced emissions into the atmosphere.

Altamirano-Fernández et al. (2022) developed a mathematical model to understand carbon capture dynamics in rapidly expanding forest covers, highlighting the importance of fire control and soil fertilization in mitigating biomass loss during regeneration cycles. SDG 13 aims to reduce greenhouse gas emissions, but many countries' targets are not aligned with the ambition needed to limit global warming below 2 degrees Celsius (Rogelj et al. 2016).

Among these, recent researches have limited information on CO_2 emissions from various specific fossil fuels and human activities, Mandal et al. (2021) evaluated the impact of GHG emissions along the coastal areas using compartmental models. In order to complement the models already in the literature, this work developed a compartmental model of CO_2 emissions from specific fossil fuels and human activity.

This study unlike the above mentioned addressed the emissions of CO_2 from global perspective and not a region or a country since this a global menace. It also addressed the interactions of human and the specific individual fossil fuels thus more informative than the previous studies, which have looked at fossil fuels in general without looking at the impact of each.

The study analyzes the model using classical techniques and simulate it using an inbuilt MATLAB ODE solver. The paper is organized as follows: Section 2 is on Model formulation; Section 3 is on Model analysis and Section 4 is on Conclusion and recommendations.

2 Model formulation

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In this study, first-order nonlinear ordinary differential equations are used to create a deterministic model for the emissions of carbon (IV) oxide from fossil fuels and human activities. The model comprises of two human populations: H consisting human populations of all ages and R comprising drivers who have received training and who are at least eighteen years old.

A compartmental structure integrated the H and R population, commonly used fossil fuels; diesel (D), petrol (P), liquid petroleum gases (L) and kerosene to represent others (O), plants (F) and carbon (IV) oxide (C).

The model was examined for positivity, boundedness, equilibrium points, and their stabilities using the next-generation matrix approach, Lyapunov function, and bifurcation analysis. In addition to the generally recognized assumptions of a compartmental model, such as non-negative beginning conditions and homogeneity in the region of the research, the following particular assumptions were also made:

i. Advection and diffusion of CO_2 in the atmosphere is insignificant. Although they significantly influence the mean diurnal cycle of net ecosystem-atmosphere CO_2 exchange, despite their seemingly insignificant long-term impact (Yi et al. 2020).
ii. Diesel and petrol are used only by drivers who are the people trained to use locomotives.
iii. Population recruitment is mainly dependent on plants as the primary source of food.
iv. The only source of depletion of fuels is through combustion.
v. Plants recruitment is solely dependent on CO_2 without which they will be depleted because during photosynthesis, carbon dioxide is a crucial reactant that gives glucose molecules carbon and oxygen.
vi. The recruitment of CO_2 from other sources and other means of its absorptions are negligible.
vii. All humans are involved in CO_2 related activities.

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The state variables were summarized as shown in the Table 1 below.

Variable	Description	Quantity	Citation
H	Human population	8 billion people	Dawson and Zhang (2024)
R	Drivers population	1360000000	Shwartz (2021)
F	Plants	4.06 billion hectares	Essl et al. (2020)
O	kerosene	20 billion gallons	Cronshaw and Mark (2021)
L	Liquid petroleum gases	1.3 billion barrels	Ekejiuba (2018)
P	Petrol	102.5 million barrels per day	Rasheed et.al. (2022)
D	Diesel	102.3 million barrels per day	Ahmad and Zhang (2020)
C	Carbon(iv) oxide	421.86 parts per million (ppm)	Iz (2022)

The parameters were summarized in the Table 2 below.

Parameter	Description	Value	Citation
ω_1	Consumption rate of plants by humans	0.3941	Lowder et al. (2021)
ω_2	Recruitment rate of humans that depends on plants for survival	0.017299	Pison (2019)
ω_3	Production rate of CO_2 through respiration of humans	0.05234	Cai et al. (2022)
γ_1	Consumption rate of kerosene through interaction of drivers and kerosene	0.0475	Lam et al. (2012)
γ_2	Production rate of CO_2 through interaction of drivers and kerosene	0.05914	Pierre (2023)

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α_1	Consumption rate of kerosene through interaction of humans and kerosene	0.0475	Lam et al. (2012)
α_2	Production rate of CO_2 through interaction of humans and kerosene	0.1165	Lam et al. (2018)
ϕ_1	Consumption rate of liquid petroleum gases through interaction of humans and liquid petroleum gases	0.001904	Cook and Mark (2021)
ϕ_2	Production rate of CO_2 through interaction of humans and liquid petroleum gases	0.5065	Singh et al. (2014)
ε_1	Consumption rate of liquid petroleum gases through interaction of drivers and liquid petroleum gases	0.000323	BP British Petroleum (2014)
ε_2	Production rate of CO_2 through interaction of drivers and liquid petroleum gases	0.08611	Holechek et al. (2022)
θ_1	Consumption rate of petrol through interaction of drivers and petrol	0.6958	Griffiths et al. (2022)
θ_2	Production rate of CO_2 through interaction of drivers and petrol	0.1711	Nguyen and Gonzalez (2021)
β_1	Consumption rate of diesel through interaction of drivers and diesel	0.1788	Holechek et al. (2022)
β_2	Production rate of CO_2 through interaction of drivers and diesel	0.1999	Petroleum (2001)
π_1	Recruitment rate of humans that depends on other means of survival apart from plants	0.0069196	Pison (2019)
π_2	Recruitment rate of plants	0.001352	Watch (2018)
π_3	Recruitment rate of kerosene, liquid petroleum gases, petrol and diesel	0.02327	Traver et al. (2022)

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ρ_1	Absorption rate of CO_2 by plants	0.1109	UNESCO (2021)
ρ_2	Recruitment rate of plants due to availability of CO_2 for photosynthesis	0.0291 9	Estoque et al. (2021)
δ_1	Destruction rate of plants through interaction of humans and plants	0.000307	Estoque et al. (2021)
δ_2	Production rate of CO_2 through interaction of humans and plants	0.2494	Marques (2020)
η	Recruitment rate of humans due to revocation and retirement of drivers	0.0002 112	Shwartz (2021)
λ	Rate of training and recruitment of drivers	0.17	Shwartz (2021)
μ_1	Natural death rate of humans and drivers	0.007625	O'Sullivan (2023)
μ_2	Natural death rate of plants	1.5%	Etzold et al. (2019)
K_1	Carrying capacity of the world	8 -16 billion people	Planet (2012)
M	Capacity of crude oil to be refined	Over 100 million barrels per day	Carlson (2023)
K_2	Carrying capacity of Kerosene, liquid petroleum gases, petrol and diesel that are in the world.	1,650,5 85,140, 000 bar rels	BP British Petroleum (2014)
a_1	Rate of refining kerosene	5.8%	Sotoodeh (2024)
a_2	Rate of refining liquid petroleum gases	2%	Sotoodeh (2024)
a_3	Rate of refining petrol	42.7%	Sotoodeh (2024)

The model framework was as follows

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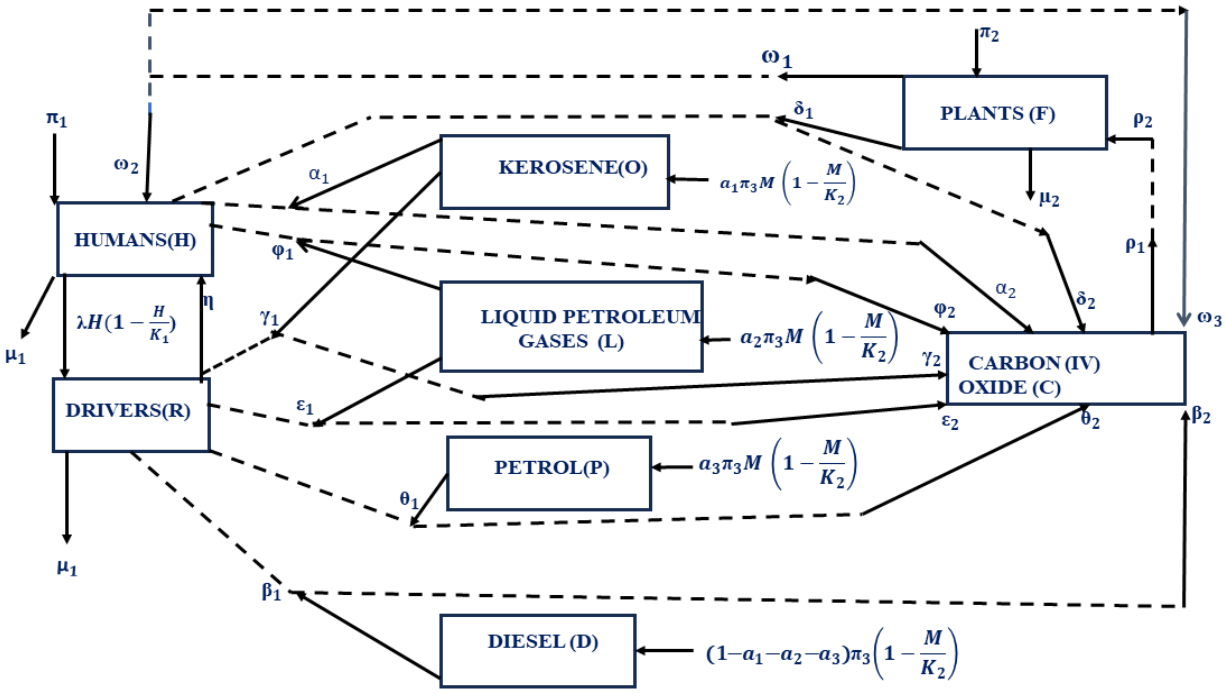

The equations derived from model flow chart were as follows;

1. $$\frac{dH}{dt} = \pi_1 + \omega_2 HF + \eta R - \mu_1 H - \lambda H\left(1 - \frac{H}{K_1}\right)$$
2. $$\frac{dR}{dt} = \lambda H\left(1 - \frac{H}{K_1}\right) - \eta R - \mu_1 R$$
3. $$\frac{dF}{dt} = \pi_2 + \rho_2 FC - \omega_1 HF - \delta_1 HF - \mu_2 F$$
4. $$\frac{dO}{dt} = a_1\pi_3 M\left(1 - \frac{M}{K_2}\right) - \alpha_1 HO - \gamma_1 RO$$
5. $$\frac{dL}{dt} = a_2\pi_3 M\left(1 - \frac{M}{K_2}\right) - \phi_1 HL - \varepsilon_1 RL$$
6. $$\frac{dP}{dt} = a_3\pi_3 M\left(1 - \frac{M}{K_2}\right) - \theta_1 RP$$
7. $$\frac{dD}{dt} = (1 - a_1 - a_2 - a_3)\pi_3 M\left(1 - \frac{M}{K_2}\right) - \beta_1 RD$$
8. $$\frac{dC}{dt} = \omega_3 HF + \alpha_2 HO + \phi_2 HL + \theta_2 RP + \beta_2 RD + \delta_2 HF + \varepsilon_2 RL + \gamma_2 RO - \rho_1 FC$$

3 Methodology

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This section is organized as follows: 3.1. Model analysis and 3.2 Model simulation.

3.1 Model analysis

This sub-section is organized as follows: 3.1.1 Positivity, 3.1.2. Boundedness of the solution, 3.1.3 Steady states, 3.1.4 Local stability, 3.1.5 Global stability and 3.1.6 Bifurcation analysis

3.1.1 Positivity

Since the model considers human population, plants, fossil fuels and carbon (iv) oxide it is necessary to show that all its state variables are positive for all future time, t Mandal et al. (2021).

Theorem 1. Let $V = \{H(t), R(t), O(t), L(t), P(t), D(t), F(t) \text{and } C(t)\} \in R_+^8$, for $H(0) \geq 0, R(0) \geq 0, O(0) \geq 0, L(0) \geq 0, P(0) \geq 0, D(0) \geq 0, F(0) \geq 0, C(0) \geq 0\}$; then the solutions of {H, R, O, L, P, } are non -negative for $t \geq 0$

Proof. The prove of the theorem follows that,

From the first differential equation (1), we have

$$\frac{dH}{dt} = \pi_1 + \omega_2 HF + \eta R - \mu_1 H - \lambda H\left(1 - \frac{H}{K_1}\right)$$

By inspection method the terms, $\pi_1 \geq 0$, $\omega_2 HF \geq 0$ and $\eta R \geq 0$.

Thus $\frac{dH}{dt} \geq - \mu_1 H - \lambda H\left(1 - \frac{H}{K_1}\right)$

By separation of variables $\frac{dH}{H\left(\mu_1 + \lambda\left(1 - \frac{H}{K_1}\right)\right)} \geq -dt$

On integrating by using partial fractions; $\int \frac{dH}{H\left(\mu_1 + \lambda\left(1 - \frac{H}{K_1}\right)\right)} \geq \int_0^t - dt$

$$H(t) \geq \frac{- K_1(\lambda + \mu_1)e^{-t(\lambda+\mu_1)+C}}{1 - \lambda e^{-t(\lambda+\mu_1)+C}}$$

Applying initial conditions at t=0, becomes,

$$H(0) \geq \frac{- K_1(\lambda + \mu_1)e^{C}}{1 - \lambda e^{C}}$$

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$$\frac{H(0)}{-K_1(\lambda+\mu_1)+\lambda H(0)} \geq e^{C}$$

$$H(t) \geq \frac{(-K_1(\lambda+\mu_1)e^{-t(\lambda+\mu_1)})(\frac{H(0)}{-K_1(\lambda+\mu_1)+\lambda H(0)})}{1-\lambda e^{-t}(\frac{H(0)}{-K_1(\lambda+\mu_1)+\lambda H(0)})}$$

$$H(t) \geq \frac{(-K_1(\lambda+\mu_1)e^{-t(\lambda+\mu_1)})H(0)}{-K_1(\lambda+\mu_1)+\lambda H(0)-\lambda e^{-t}}$$

By inspection, the numerator is negative. Since $H \leq K_1$, $\lambda(H(0)-K_1)$ is negative thus the denominator is also negative hence $H(t) \geq 0$.

Similarly, from equation 2, it follows

$$R(t) \geq R(0)\exp(-(\mu_1+\eta)t) \geq 0$$

From the first differential equation (3), we have

$$\frac{dF}{dt} = \pi_2 + \rho_2 FC - \omega_1 HF - \delta_1 HF - \mu_2 F$$

By inspection method the terms, $\pi_2 \geq 0$ and $\rho_2 FC \geq 0$

$$\frac{dF}{dt} \geq -\omega_1 HF - \delta_1 HF - \mu_2 F$$

$$\frac{dF}{F} \geq -(\omega_1 H + \delta_1 H + \mu_2)dt$$

$$F(t) \geq e^{c} e^{-\int_0^t(\omega_1 H(a)+\delta_1 H(a)+\mu_2)da}$$

Applying initial conditions at t=0, becomes,

$$F(0) \geq e^{c}$$

$$F(t) \geq F(0)e^{-\int_0^t(\omega_1 H(a)+\delta_1 H(a)+\mu_2)da}$$

Since $(0) \geq 0$ and $e^{-\int_0^t(\omega_1 H(a)+\delta_1 H(a)+\mu_2)da} \geq 0$, exponential of a negative is always positive hence $F(t) \geq 0$.

From the first differential equation (4), we have

$$\frac{dO}{dt} = a_1\pi_3 M\left(1-\frac{M}{K_2}\right) - \alpha_1 HO - \gamma_1 RO$$

By inspection method the terms, $a_1\pi_3 M\left(1-\frac{M}{K_2}\right) \geq 0$.

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$$\frac{dO}{dt} \geq -O(\alpha_1 H + \gamma_1 R)$$

$$\ln O \geq -(\alpha_1 H(a) + \gamma_1 R(a))da + c$$

$$O(t) \geq e^{c} e^{-\int_0^t (\alpha_1 H(a) + \gamma_1 R(a))da}$$

Applying initial conditions at t=0, becomes,

$$O(0) \geq e^{c}$$

$$O(t) \geq O(0) e^{-\int_0^t (\alpha_1 H(a) + \gamma_1 R(a))da} \geq 0$$

Similarly, from equations 5-7, it follows

$$\frac{dL}{dt} = a_2 \pi_3 M\left(1 - \frac{M}{K_2}\right) - \phi_1 HL - \varepsilon_1 RL$$

By inspection method the terms, $a_2 \pi_3 M\left(1 - \frac{M}{K_2}\right) \geq 0$.

$$\frac{dL}{dt} \geq -L(\phi_1 H + \varepsilon_1 R)$$

$$\frac{dL}{L} \geq -(\phi_1 H + \varepsilon_1 R)dt$$

$$L(t) \geq e^{c} e^{-\int_0^t (\phi_1 H(a) + \varepsilon_1 R(a))da}$$

Applying initial conditions at t=0, becomes,

$$L(0) \geq e^{c}$$

$$L(t) \geq L(0) e^{-\int_0^t (\phi_1 H(a) + \varepsilon_1 R(a))da} \geq 0$$

$$\frac{dP}{dt} = a_3 \pi_3 M\left(1 - \frac{M}{K_2}\right) - \theta_1 RP$$

By inspection method the terms, $a_3 \pi_3 M\left(1 - \frac{M}{K_2}\right) \geq 0$.

$$\frac{dP}{dt} \geq -\theta_1 RP$$

$$\frac{dP}{P} \geq -\theta_1 R dt$$

$$P(t) \geq e^{c} e^{-\int_0^t ((\theta_1 R(a))da}$$

Applying initial conditions at t=0, becomes,

$$P(0) \geq e^{c}$$

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$$P(t) \geq P(0)e^{-\int_0^t((\theta_1 R(a))da} \geq 0$$

$$\frac{dD}{dt} = (1 - a_1 - a_2 - a_3)\pi_3 M\left(1 - \frac{M}{K_2}\right) - \beta_1 RD$$

By inspection method the terms, $(1 - a_1 - a_2 - a_3)\pi_3 M\left(1 - \frac{M}{K_2}\right) \geq 0$.

$$\frac{dD}{D} \geq -\beta_1 R dt$$

$$D(t) \geq e^c e^{-\int_0^t((\beta_1 R(a))da}$$

Applying initial conditions at t=0, becomes,

$$D(0) \geq e^c$$

$$D(t) \geq D(0)e^{-\int_0^t((\beta_1 R(a))da} \geq 0$$

$$\frac{dC}{dt} = \omega_3 HF + \alpha_2 HO + \phi_2 HL + \theta_2 RP + \beta_2 RD + \delta_2 HF + \varepsilon_2 RL + \gamma_2 RO - \rho_1 FC$$

By inspection method the terms, $\omega_3 HF + \alpha_2 HO + \phi_2 HL + \theta_2 RP + \beta_2 RD + \delta_2 HF + \varepsilon_2 RL + \gamma_2 RO \geq 0$.

$$\frac{dC}{C} \geq -\rho_1 F dt$$

$$C(t) \geq e^c e^{-\int_0^t((\rho_1 F(a))da}$$

Applying initial conditions at t=0, becomes,

$$C(0) \geq e^c$$

$$C(t) \geq C(0)e^{-\int_0^t((\rho_1 R(a))da} \geq 0$$

The theorem is proved.

3.1.2 Boundedness of the solution

The boundedness of a solution in a mathematical model is a property that indicates whether the values of the decision variables are restricted within a certain range (Otieno 2022). For the system to be mathematically realistic and practical, it is necessary to show that its state variables are positive and bounded for all t. That is, the solution of the system with a positive initial value will remain positive for all $t \geq 0$.

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$N(t) = H(t) + R(t)$

By differentiating, the above equation gives,

$$\frac{dN}{dt} = \frac{dH}{dt} + \frac{dR}{dt}$$

$$\frac{dN}{dt} = \pi_1 + \omega_2 HF + \eta R - \mu_1 H - \lambda H\left(1 - \frac{H}{K_1}\right) + \lambda H\left(1 - \frac{H}{K_1}\right) - \mu_1 R - \eta R$$

$$\frac{dN}{dt} = \pi_1 + \omega_2 HF - \mu_1 N$$

$$\frac{dN}{dt} + \mu_1 N \leq \pi_1$$

Integrating both sides of (above), we have

$N(t) \leq Ae^{\mu_1 t}$ $\qquad N_P \leq B$

$\frac{dN}{dt} + \mu_1 B \leq \pi_1$ $\qquad$ When $\frac{dN}{dt} = 0$,

$$B \leq \frac{\pi_1}{\mu_1},$$

$$N(t) \leq Ae^{\mu_1 t} + \frac{\pi_1}{\mu_1}$$

Applying the initial conditions at t=0 then, N= N (0)

$N(0) \leq A + \frac{\pi_1}{\mu_1}$, $\qquad N(0) - \frac{\pi_1}{\mu_1} \leq A$

Substituting,

$$N(t) \leq (N(0) - \frac{\pi_1}{\mu_1})e^{\mu_1 t} + \frac{\pi_1}{\mu_1}$$

Introducing limits,

$$\lim_{t\to\infty} N(t) \leq \frac{\pi_1}{\mu_1}$$

Implying that,

$$0 \leq N(t) \leq \frac{\pi_1}{\mu_1}$$

Also, $M(t) = O(t) + L(t) + P(t) + D(t)$

$$\frac{dM}{dt} = \frac{dO}{dt} + \frac{dL}{dt} + \frac{dP}{dt} + \frac{dD}{dt}.$$

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$$\frac{dM}{dt} = a_1\pi_3 M\left(1 - \frac{M}{K_2}\right) - \alpha_1 HO - \gamma_1 RO + a_2\pi_3 M\left(1 - \frac{M}{K_2}\right) - \phi_1 HL - \varepsilon_1 RL + a_3\pi_3 M\left(1 - \frac{M}{K_2}\right)$$

$$- \theta_1 RP + (1 - a_1 - a_2 - a_3)\pi_3 M\left(1 - \frac{M}{K_2}\right) - \beta_1 RD$$

$$\frac{dM}{dt} \leq \pi_3 M\left(1 - \frac{M}{K_2}\right)$$

By separating of variables and integrating, we have

$$\frac{dM}{M\left(1 - \frac{M}{K_2}\right)} \leq \pi_3 dt$$

$$M(t) \leq K_2 - e^{-\pi_3 t + c}$$

On applying initial conditions, t=0, M=M(0);

$$M(0) \leq K_2 - e^{c}$$

$$e^{c} \leq K_2 - M(0)$$

Therefore,

$$M(t) \leq K_2 - e^{-\pi_3 t}(K_2 - M(0))$$

$$\lim_{t\to\infty} M(t) \leq K_2$$

Therefore;

$$M(t) \leq K_2$$

Implying that, $0 \leq M(t) \leq K_2$

Similarly, $F(t) = F(t)$

$$\frac{dF}{dt} = \pi_2 + \rho_2 FC - \omega_1 HF - \delta_1 HF - \mu_2 F$$

$$\frac{dF}{dt} + \mu_2 F \leq \pi_2$$

Integrating both sides of (above) using the separation of variables of inequality, we have

$$F(t) \leq Ae^{\mu_2 t}, \qquad F_P \leq B$$

$$\frac{dF}{dt} + \mu_2 B \leq \pi_2 \qquad \text{When} \quad \frac{dF}{dt} = 0, \quad B \leq \frac{\pi_2}{\mu_2},$$

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$$F(t) \leq Ae^{\mu_2 t} + \frac{\pi_2}{\mu_2}$$

Applying the initial conditions at t=0 then, F= F (0)

$$F(0) \leq A + \frac{\pi_2}{\mu_2}, \qquad F(0) - \frac{\pi_2}{\mu_2} \leq A$$

Substituting,

$$F(t) \leq (F(0) - \frac{\pi_2}{\mu_2})e^{\mu_2 t} + \frac{\pi_2}{\mu_2}$$

Introducing limits,

$$\lim_{t\to\infty} F(t) \leq \frac{\pi_2}{\mu_2}$$

Implying that,

$$0 \leq F(t) \leq \frac{\pi_2}{\mu_2}$$

3.1.3 Steady states

In the analysis of a system of ordinary differential equations, the state variables that do not change with time are more necessarily to be considered (Odhiambo 2023). That is there is no change of situation in the system. These solutions are known to as the equilibrium points. Thus, the steady states are obtained by equating system of equations in (1-8) to zero. Thus, for the sake of studying the stability of the proposed model 1, the equilibrium points of the system required to be determined

Case 1: No humans, drivers, kerosene, liquid petroleum gases, petrol, diesel, plants and carbon (IV) oxide exist.

This implies that $H = R = F = O = L = P = D = C \neq 0$. The equilibrium point is then denoted as E^0.Thus if the nothing exist then the equilibrium point is obtained by setting the variables H^0, R^0, F^0, O^0, L^0, P^0, D^0 and C^0 to zero. Consequently, the right-hand side of equations (1-8) are set to zero to obtain a trivial solution, which is $E^0 = (0,0,0,0,0,0,0,0)$

Case 2: In the absence of humans and their related activities.

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The equilibrium points where there are no human activities is denoted by E^1. In such a case the variables $H^1=0$, $R^1=0$, $F^1 \neq 0$, $O^1 \neq 0$, $L^1 \neq 0$, $P^1 \neq 0$, $D^1 \neq 0$ and $C^1 \neq 0$. As a result, equations (1) and (2) are reduced to zero.

$$\pi_2 + \rho_2 F^1 C^1 - F^1 = 0$$

$$F^1 = \frac{\pi_2}{\rho_2 C^1 - \mu_2}$$

$$a_1 \pi_3 M^1 \left(1 - \frac{M^1}{K_2}\right) = 0,\ M^1 = K_2$$

$$a_2 \pi_3 M^1 \left(1 - \frac{M^1}{K_2}\right) = 0\ ,\ M^1 = K_2$$

$$a_3 \pi_3 M^1 \left(1 - \frac{M^1}{K_2}\right) = 0,\ M^1 = K_2$$

$$(1 - a_1 - a_2 - a_3) \pi_3 M^1 \left(1 - \frac{M^1}{K_2}\right) = 0, M^1 = K_2$$

$$\omega_3 H^1 F^1 + \alpha_2 H^1 O^1 + \phi_2 H^1 L^1 + \theta_2 R^1 P^1 + \beta_2 R^1 D^1 + \delta_2 H^1 F^1 + \varepsilon_2 R^1 L^1 + \gamma_2 R^1 O^1 - \rho_1 F^1 C^1 = 0$$

$$C^1 = \frac{\omega_3 H^1 F^1 + \alpha_2 H^1 O^1 + \phi_2 H^1 L^1 + \theta_2 R^1 P^1 + \beta_2 R^1 D^1 + \delta_2 H^1 F^1 + \varepsilon_2 R^1 L^1 + \gamma_2 R^1 O^1}{\rho_1 F^1}$$

$$E^1 = (0,0,F^1,M^1,M^1,M^1,M^1,C^1)$$

Case 3: In the absence fossil fuels.

The equilibrium points where there are no fossil fuels is denoted by E^2. In such a case the variables $H^2 \neq 0$, $R^2 \neq 0$, $F^2 \neq 0$, $O^2 = 0$, $L^2 = 0$, $P^2 = 0$, $D^2 = 0$ and $C^2 \neq 0$. As a result, equations (1), (2) and (8) are reduced to zero.

$$\pi_1 + \omega_2 H^2 F^2 + \eta R^2 - \mu_1 H^2 - \lambda H^2 \left(1 - \frac{H^2}{K_1}\right) = 0$$

$$H^2 = \frac{K_1 \left(\lambda - F\omega_2 - \sqrt{-\frac{4\lambda(R^2\eta + \pi_1)}{K_1} + (-\lambda + F^2\omega_2 - \mu_1)^2} + \mu_1\right)}{2\lambda} \text{ or}$$

$$\frac{K_1(\lambda - F^2\omega_2 + \sqrt{-\frac{4\lambda(R^2\eta + \pi_1)}{K_1} + (-\lambda + F^2\omega_2 - \mu_1)^2} + \mu_1)}{2\lambda}\}\}$$

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$$\lambda H^2\left(1 - \frac{H^2}{K_1}\right) - \eta R^2 - \mu_1 R^2 = 0$$

$$R^2 = \frac{\lambda H^2\left(1 - \frac{H^2}{K_1}\right)}{\eta + \mu_1}$$

$$\pi_2 + \rho_2 F^2 C^2 - \omega_1 H^2 F^2 - \delta_1 H^2 F^2 - \mu_2 F^2 = 0$$

$$F^2 = \frac{\pi_2}{H^2\delta_1 - C^2\rho_2 + H^2\omega_1 + \mu_2}$$

$$\omega_3 H^2 F^2 + \delta_2 H^2 F^2 - \rho_1 F^2 C^2 = 0$$

$$C^2 = \frac{H^2\delta_2 + H^2\omega_3}{\rho_1}$$

$$E^2 = (H^2, R^2, F^2, 0, 0, 0, 0, C^2)$$

Case 4: In the absence drivers.

The equilibrium points where there are no drivers is denoted by E^3. In such a case the variables $H^3 \neq 0$, $R^3 = 0$, $F^3 \neq 0$, $O^3 \neq 0$, $L^3 \neq 0$, $P^3 = 0$, $D^3 = 0$ and $C^3 \neq 0$. As a result, equations (1), (2) and (8) are reduced to zero.

$$\pi_1 + \omega_2 H^3 F^3 - \mu_1 H^3 = 0$$

$$H^3 = -\frac{\pi_1}{F^3\omega_2 - \mu_1}$$

$$\pi_2 + \rho_2 F^3 C^3 - \omega_1 H^3 F^3 - \delta_1 H^3 F^3 - \mu_2 F^3 = 0$$

$$F^3 = \frac{\pi_2}{H^3\delta_1 - C^3\rho_2 + H^3\omega_1 + \mu_2}$$

$$a_1\pi_3 M^3\left(1 - \frac{M^3}{K_2}\right) - \alpha_1 H^3 O^3 = 0$$

$$O^3 = \frac{M^3[1 - \frac{M^3}{K_2}]a_1\pi_3}{H^3\alpha_1}$$

$$a_2\pi_3 M^3\left(1 - \frac{M^3}{K_2}\right) - \phi_1 H^3 L^3 = 0$$

$$L^3 = \frac{M^3[1 - \frac{M^3}{K_2}]a_2\pi_3}{H^3\phi_1}$$

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$$\omega_3 H^3 F^3 + \alpha_2 H^3 O^3 + \phi_2 H^3 L^3 + \delta_2 H^3 F^3 - \rho_1 F^3 C^3 = 0$$

$$C^3 = \frac{H^3 O^3 \alpha_2 + F^3 H^3 \delta_2 + H^3 L^3 \phi_2 + F^3 H^3 \omega_3}{F^3 \rho_1}$$

$$E^3 = (H^3, 0, F^3, O^3, L^3, 0, 0, C^3)$$

Case 5: Coexistence in the system

For the existence of the eight categories, the study established many equilibrium points including complex equilibrium points. However, complex equilibrium points are not included because the system is defined in the positive region and they are not tractable. In such a case the variables $H^4 \neq 0$, $R^4 \neq 0$, $F^4 \neq 0$, $O^4 \neq 0$, $L^4 \neq 0$, $P^4 \neq 0$, $D^4 \neq 0$ and $C^4 \neq 0$.

$$\pi_1 + \omega_2 H^4 F^4 + \eta R^4 - \mu_1 H^4 - \lambda H^4 \left(1 - \frac{H^4}{K_1}\right) = 0$$

$$H^4 = \frac{K_1 \left(\lambda - F^4 \omega_2 - \sqrt{-\frac{4\lambda(R^4 \eta + \pi_1)}{K_1} + (-\lambda + F^4 \omega_2 - \mu_1)^2} + \mu_1\right)}{2\lambda} \text{ or}$$

$$\frac{K_1 (\lambda - F^4 \omega_2 + \sqrt{-\frac{4\lambda(R^4 \eta + \pi_1)}{K_1} + (-\lambda + F^4 \omega_2 - \mu_1)^2} + \mu_1)}{2\lambda}$$

$$\lambda H^4 \left(1 - \frac{H^4}{K_1}\right) - \eta R^4 - \mu_1 R^4 = 0$$

$$R^4 = \frac{\lambda H^4 \left(1 - \frac{H^4}{K_1}\right)}{\eta + \mu_1}$$

$$\pi_2 + \rho_2 F^4 C^4 - \omega_1 H^4 F^4 - \delta_1 H^4 F^4 - \mu_2 F^4 = 0$$

$$F^4 = \frac{\pi_2}{H^4 \delta_1 - C^4 \rho_2 + H^4 \omega_1 + \mu_2}$$

$$a_1 \pi_3 M^4 \left(1 - \frac{M^4}{K_2}\right) - \alpha_1 H^4 O^4 - \gamma_1 R^4 O^4 = 0$$

$$O^4 = \frac{M^4 [1 - \frac{M^4}{K_2}] a_1 \pi_3}{H^4 \alpha_1 + R^4 \gamma_1}$$

$$a_2 \pi_3 M^4 \left(1 - \frac{M^4}{K_2}\right) - \phi_1 H^4 L^4 - \varepsilon_1 R^4 L^4 = 0$$

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$$L^4 = \frac{M^4[1 - \frac{M^4}{K_2}]a_2\pi_3}{R^4 + H^4\phi_1}$$

$$a_3\pi_3M^4\left(1 - \frac{M^4}{K_2}\right) - \theta_1R^4P^4 = 0$$

$$P^4 = \frac{M^4[1 - \frac{M^4}{K_2}]a_3\pi_3}{R^4\theta_1}$$

$$(1 - a_1 - a_2 - a_3)\pi_3M^4\left(1 - \frac{M^4}{K_2}\right) - \beta_1R^4D^4 = 0$$

$$D^4 = \frac{M^4(M^4 - K_2)\pi_3(-1 + a_1(-a - a_{23}))}{R^4K_2\beta_1}$$

$$\omega_3H^4F^4 + \alpha_2H^4O^4 + \phi_2H^4L^4 + \theta_2R^4P^4 + \beta_2R^4D^4 + \delta_2H^4F^4 + \varepsilon_2R^4L^4 + \gamma_2R^4O^4 - \rho_1F^4C^4 = 0$$

$$C^4 = \frac{\varepsilon_2R^4L^4 + \alpha_2H^4O^4 + \beta_2R^4D^4 + \gamma_2R^4O^4 + \delta_2H^4F^4 + \theta_2R^4P^4 + \phi_2H^4L^4 + \omega_3H^4F^4}{\rho_1F^4}$$

$$E^4 = (H^4,R^4,F^4,O^4,L^4,P^4,D^4,C^4)$$

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3.1.4 Local stability

The local stability of any system is determined by the nature of the eigenvalues of the Jacobian matrix, if eigenvalues are negative, it is stable otherwise unstable. In our case where the matrix is large such that the nature of eigenvalues cannot be easily determined, we will apply Routh–Hurwitz criteria to determine stability.

The study therefore considered the steady states of the Jacobian matrix below under different cases as discussed below.

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$$\begin{vmatrix} -\lambda(1-\frac{H}{K_1})+\frac{H\lambda}{K_1}+F\omega_2-\mu_1 & \eta & H\omega_2 & 0 & 0 & 0 \\ \lambda(1-\frac{H}{K_1})-\frac{H\lambda}{K_1} & -\eta-\mu_1 & 0 & 0 & 0 & 0 \\ -F\delta_1-F\omega_1 & 0 & \pi_2-H\delta_1+C\rho_2-H\omega_1-\mu_2 & 0 & 0 & 0 \\ -O\alpha_1 & -O\gamma_1 & 0 & -H\alpha_1-R\gamma_1 & 0 & 0 \\ -L\phi_1 & -L & 0 & 0 & -R-H\phi_1 & 0 \\ 0 & -P\theta_1 & 0 & 0 & 0 & -R\theta_1 \\ 0 & -D\beta_1 & 0 & 0 & 0 & 0 \\ O\alpha_2+F\delta_2+L\phi_2+F\omega_3 & L+D\beta_2+O\gamma_2+P\theta_2 & H\delta_2-C\rho_1+H\omega_3 & H\alpha_2+R\gamma_2 & R+H\phi_2 & R\theta_2 \end{vmatrix}$$

Substitute each of the steady states and determine the eigen values of the matrix and establish the necessary conditions to make it negative. Using the above Jacobian matrix A, to obtain the Eigen values $\Box$ we find the determinant of $(A - \Box I)$. The characteristic polynomial of the above matrix is given as;

$$a_1\Box^8 + a_2\Box^7 + a_3\Box^6 + a_4\Box^5 + a_5\Box^4 + a_6\Box^3 + a_7\Box^2 + a_8\Box + a_9 = 0$$

By Routh-Hurwitz criteria for stability, the system (1) – (8) is locally asymptomatically stable at the equilibrium point (E^0) if and only if $\Box_1 > 0$, $\Box_2 > 0$, $b_1 > 0$, $c_1 > 0$, $d_1 > 0$, $\Box_1 > 0$, $a_7 > 0$, $a_8 > 0$ $\Box\Box\Box$ $a_9 > 0$ are satisfied and otherwise unstable.

Theorem 1. The trivial equilibrium point, E^0, is stable if the eigenvalues of the variational matrix are all less than zero. Otherwise, the system will be unstable.

Proof. The variational matrix evaluated at E^0 is given by

$$\begin{vmatrix} -\lambda-\mu_1 & \eta & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & -\eta-\mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi_2-\mu_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Where eigenvalues are $\Box_1 = -\lambda - \mu_1$, $\Box_2 = -\eta - \mu_1$ and $\Box_3 = \pi_2 - \mu_2$.

The equilibrium point E^0 is stable if and only if $\lambda < -\mu_1$, $\eta < -\mu_1$ and $\pi_2 < -\mu_2$. Otherwise, the system will be unstable.

Theorem 2. The equilibrium point E^1 where the variables $H^1=0$, $R^1=0$, $F^1 \neq 0$, $O^1 \neq 0$, $L^1 \neq 0$, $P^1 \neq 0$, $D^1 \neq 0$ and $C^1 \neq 0$ by Routh–Hurwitz criterion the coefficients of the characteristic polynomial are

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$a_1^1 = 1$, $a_2^1 = \eta + \lambda - \pi_2 + F\rho_1 - C\rho_2 - F\omega_2 + 2\mu_1 + \mu_2$, $a_3^1 = -\eta\pi_2 - \lambda\pi_2 + F\eta\rho_1 + F\lambda\rho_1 - F\pi_2\rho_1 - C\eta\rho_2 - C\lambda\rho_2 - F\eta\omega_2 + F\pi_2\omega_2 - F^2\rho_1\omega_2 + CF\rho_2\omega_2 + \eta\mu_1 + \lambda\mu_1 - 2\pi_2\mu_1 + 2F\rho_1\mu_1 - 2C\rho_2\mu_1 - F\omega_2\mu_1 + \mu_1^2 + \eta\mu_2 + \lambda\mu_2 + F\rho_1\mu_2 - F\omega_2\mu_2 + 2\mu_1\mu_2$

$$a_4^1 = -F\eta\pi_2\rho_1 - F\lambda\pi_2\rho_1 + F\eta\pi_2\omega_2 - F^2\eta\rho_1\omega_2 + F^2\pi_2\rho_1\omega_2 + CF\eta\rho_2\omega_2 - \eta\pi_2\mu_1 - \lambda\pi_2\mu_1 + F\eta\rho_1\mu_1 + F\lambda\rho_1\mu_1 - 2F\pi_2\rho_1\mu_1 - C\eta\rho_2\mu_1 - C\lambda\rho_2\mu_1 + F\pi_2\omega_2\mu_1 - F^2\rho_1\omega_2\mu_1 + CF\rho_2\omega_2\mu_1 - \pi_2\mu_1^2 + F\rho_1\mu_1^2 - C\rho_2\mu_1^2 + F\eta\rho_1\mu_2 + F\lambda\rho_1\mu_2 - F\eta\omega_2\mu_2 - F^2\rho_1\omega_2\mu_2 + \eta\mu_1\mu_2 + \lambda\mu_1\mu_2 + 2F\rho_1\mu_1\mu_2 - F\omega_2\mu_1\mu_2 + \mu_1^2\mu_2$$

$$a_5^1 = F^2\eta\pi_2\rho_1\omega_2 - F\eta\pi_2\rho_1\mu_1 - F\lambda\pi_2\rho_1\mu_1 + F^2\pi_2\rho_1\omega_2\mu_1 - F\pi_2\rho_1\mu_1^2 - F^2\eta\rho_1\omega_2\mu_2 + F\eta\rho_1\mu_1\mu_2 + F\lambda\rho_1\mu_1\mu_2 - F^2\rho_1\omega_2\mu_1\mu_2 + F\rho_1\mu_1^2\mu_2$$

$a_6^1 = 0$, $a_7^1 = 0$, $a_8^1 = 0$ and $a_9^1 = 0$.

Theorem 3. The equilibrium point E^2 where the variables $H^2 \neq 0$, $R^2 \neq 0$, $F^2 \neq 0$, $O^2 = 0$, $L^2 = 0$, $P^2 = 0$, $D^2 = 0$ and $C^2 \neq 0$ by Routh–Hurwitz criterion the coefficients of the characteristic polynomial are

$a_1^2 = 1$, $a_2^2 = \eta + \lambda - \pi_2 + F\rho_1 - C\rho_2 - F\omega_2 + 2\mu_1 + \mu_2$, $a_3^2 = -\eta\pi_2 - \lambda\pi_2 + F\eta\rho_1 + F\lambda\rho_1 - F\pi_2\rho_1 - C\eta\rho_2 - C\lambda\rho_2 - F\eta\omega_2 + F\pi_2\omega_2 - F^2\rho_1\omega_2 + CF\rho_2\omega_2 + \eta\mu_1 + \lambda\mu_1 - 2\pi_2\mu_1 + 2F\rho_1\mu_1 - 2C\rho_2\mu_1 - F\omega_2\mu_1 + \mu_1^2 + \eta\mu_2 + \lambda\mu_2 + F\rho_1\mu_2 - F\omega_2\mu_2 + 2\mu_1\mu_2$,

$$a_4^2 = -F\eta\pi_2\rho_1 - F\lambda\pi_2\rho_1 + F\eta\pi_2\omega_2 - F^2\eta\rho_1\omega_2 + F^2\pi_2\rho_1\omega_2 + CF\eta\rho_2\omega_2 - \eta\pi_2\mu_1 - \lambda\pi_2\mu_1 + F\eta\rho_1\mu_1 + F\lambda\rho_1\mu_1 - 2F\pi_2\rho_1\mu_1 - C\eta\rho_2\mu_1 - C\lambda\rho_2\mu_1 + F\pi_2\omega_2\mu_1 - F^2\rho_1\omega_2\mu_1 + CF\rho_2\omega_2\mu_1 - \pi_2\mu_1^2 + F\rho_1\mu_1^2 - C\rho_2\mu_1^2 + F\eta\rho_1\mu_2 + F\lambda\rho_1\mu_2 - F\eta\omega_2\mu_2 - F^2\rho_1\omega_2\mu_2 + \eta\mu_1\mu_2 + \lambda\mu_1\mu_2 + 2F\rho_1\mu_1\mu_2 - F\omega_2\mu_1\mu_2 + \mu_1^2\mu_2$$

$$a_5^2 = F^2\eta\pi_2\rho_1\omega_2 - F\eta\pi_2\rho_1\mu_1 - F\lambda\pi_2\rho_1\mu_1 + F^2\pi_2\rho_1\omega_2\mu_1 - F\pi_2\rho_1\mu_1^2 - F^2\eta\rho_1\omega_2\mu_2 + F\eta\rho_1\mu_1\mu_2 + F\lambda\rho_1\mu_1\mu_2 - F^2\rho_1\omega_2\mu_1\mu_2 + F\rho_1\mu_1^2\mu_2$$

$a_6^2 = 0$, $a_7^2 = 0$, $a_8^2 = 0$, $a_9^2 = 0$

Theorem 4. The equilibrium point E^3 where the variables $H^3 \neq 0$, $R^3 = 0$, $F^3 \neq 0$, $O^3 \neq 0$, $L^3 \neq 0$, $P^3 = 0$, $D^3 = 0$ and $C^3 \neq 0$ by Routh–Hurwitz criterion the coefficients of the characteristic polynomial are as shown in the appendix.

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Theorem 5. The equilibrium point E^4 where the variables $H^4 \geq 0$, $R^4 \geq 0$, $F^4 \geq 0$, $O^4 \geq 0$, $L^4 \geq 0$, $P^4 \geq 0$, $D^4 \geq 0$ and $C^4 \geq 0$ by Routh–Hurwitz criterion the coefficients of the characteristic polynomial are

$a_1^4 = 1$, while a_2^4, a_3^4, a_4^4, a_6^4, a_7^4, a_8^4 and a_9^4 are shown in the appendix.

3.1.5 Global stability

A global stability is the stability of a steady state at a point away from the equilibrium point. One common approach in studying the global asymptotically stable state is by use of Lyapunov function. Linear stability analysis tells us how a system behaves near an equilibrium point. It does not however tell us anything about what happens farther away from equilibrium. For higher dimensions, we considered using a Lyapunov function. This study considered Lyapunov similar to the one proposed by (Odhiambo 2023).

Theorem 2

$V'(E^*) = 0$ where $E^* = H^*,R^*,F^*,O^*,L^*,P^*,D^*,C^*$ and $V = (H,R,F,O,L,P,D,C) > 0$. The time derivative of V is $\frac{dV}{dt} \leq 0$ implied that the system is just stable and $\frac{dV}{dt} < 0$ implies that E^* is globally asymptotically stable.

Proof

Let

$$V = \frac{(H - H^*)^2}{2} + \sigma_1 \frac{(R - R^*)^2}{2} + \sigma_2 \frac{(F - F^*)^2}{2} + \sigma_3 \frac{(O - O^*)^2}{2} + \sigma_4 \frac{(L - L^*)^2}{2} + \sigma_5 \frac{(P - P^*)^2}{2} + \sigma_6 \frac{(D - D^*)^2}{2} + \sigma_7 \frac{(C - C^*)^2}{2}$$

be the Lyapunov function, where $\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_7 > 0$ were taken properly such that the above function satisfied the Lyapunov conditions below.

$$V = \frac{(H - H^*)^2}{2} + \sigma_1 \frac{(R - R^*)^2}{2} + \sigma_2 \frac{(F - F^*)^2}{2} + \sigma_3 \frac{(O - O^*)^2}{2} + \sigma_4 \frac{(L - L^*)^2}{2} + \sigma_5 \frac{(P - P^*)^2}{2} + \sigma_6 \frac{(D - D^*)^2}{2} + \sigma_7 \frac{(C - C^*)^2}{2}$$

Getting time derivative of the above equation

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$$\frac{dV}{dt} = (H - H^*)\frac{dH}{dt} + \sigma_1(R - R^*)\frac{dR}{dt} + \sigma_2(F - F^*)\frac{dF}{dt} + \sigma_3(O - O^*)\frac{dO}{dt} + \sigma_4(L - L^*)\frac{dL}{dt} + \sigma_5(P - P^*)\frac{dP}{dt}$$
$$+ \sigma_6(D - D^*)\frac{dD}{dt} + \sigma_7(C - C^*)\frac{dC}{dt}$$

By substituting the values of $\frac{dH}{dt}, \frac{dR}{dt}, \frac{dF}{dt}, \frac{dO}{dt}, \frac{dL}{dt}, \frac{dP}{dt}, \frac{dD}{dt}, \frac{dC}{dt}$ and factorizing H,R,F,O,L,P,D and C results to

$$\frac{dV}{dt} = [(H - H^*)\{\left(\frac{\pi_1}{H} + \omega_2 F + \frac{\eta R}{H} - \mu_1 - \lambda\left(1 - \frac{H}{K_1}\right)\right)(H)\} + \sigma_1(R - R^*)\{\left(\frac{\lambda H}{R}\left(1 - \frac{H}{K_1}\right) - \eta - \mu_1\right)(R)\}$$
$$+ \sigma_2(F - F^*)\{\left(\frac{\pi_2}{F} + \rho_2 C - \omega_1 H - \delta_1 H - \mu_2\right)(F)\}$$
$$+ \sigma_3(O - O^*)\{\left(a_1\pi_3\frac{M}{O}\left(1 - \frac{M}{K_2}\right) - \alpha_1 H - \gamma_1 R\right)(O)\} + \sigma_4(L - L^*)\{\left(a_2\pi_3\frac{M}{L}\left(1 - \frac{M}{K_2}\right) - \phi_1 H - \varepsilon_1 R\right)(L)\}$$
$$+ \sigma_5(P - P^*)\{\left(a_3\pi_3\frac{M}{P}\left(1 - \frac{M}{K_2}\right) - \theta_1 R\right)(P)\} + \sigma_6(D - D^*)\{\left((1 - a_1 - a_2 - a_3)\pi_3\frac{M}{D}\left(1 - \frac{M}{K_2}\right) - \beta_1 R\right)(D)\}$$
$$+ \sigma_7(C - C^*)\left\{\left(\frac{\omega_3}{C}HF + \frac{\alpha_2}{C}HO + \frac{\phi_2}{C}HL + \frac{\theta_2}{C}RP + \frac{\beta_2}{C}RD + \frac{\delta_2}{C}HF + \frac{\varepsilon_2}{C}RL\right.\right.$$
$$\left.\left. + \frac{\gamma_2}{C}RO - \rho_1 F\right)(C)\}\right]$$

At special case $H = H^*$, $R = R^*$, $F = F^*$, $O = O^*$, $L = L^*$, $P = P^*$, $D = D^*$, $C = C^*$ are also equilibrium points, then substituting and further simplifying equation resulted to,

$$\frac{dV}{dt} = [(H - H^*)^2\{\left(\frac{\pi_1}{H} + \omega_2 F + \frac{\eta R}{H} - \mu_1 - \lambda\left(1 - \frac{H}{K_1}\right)\right)\} + \sigma_1(R - R^*)^2\{\left(\frac{\lambda H}{R}\left(1 - \frac{H}{K_1}\right) - \eta - \mu_1\right)\}$$
$$+ \sigma_2(F - F^*)^2\{\left(\frac{\pi_2}{F} + \rho_2 C - \omega_1 H - \delta_1 H - \mu_2\right)\}$$
$$+ \sigma_3(O - O^*)^2\{\left(a_1\pi_3\frac{M}{O}\left(1 - \frac{M}{K_2}\right) - \alpha_1 H - \gamma_1 R\right)\} + \sigma_4(L - L^*)^2\{\left(a_2\pi_3\frac{M}{L}\left(1 - \frac{M}{K_2}\right) - \phi_1 H - \varepsilon_1 R\right)\}$$
$$+ \sigma_5(P - P^*)^2\{\left(a_3\pi_3\frac{M}{P}\left(1 - \frac{M}{K_2}\right) - \theta_1 R\right)\} + \sigma_6(D - D^*)^2\{\left((1 - a_1 - a_2 - a_3)\pi_3\frac{M}{D}\left(1 - \frac{M}{K_2}\right) - \beta_1 R\right)\}$$
$$+ \sigma_7(C - C^*)^2\left\{\left(\frac{\omega_3}{C}HF + \frac{\alpha_2}{C}HO + \frac{\phi_2}{C}HL + \frac{\theta_2}{C}RP + \frac{\beta_2}{C}RD + \frac{\delta_2}{C}HF + \frac{\varepsilon_2}{C}RL\right.\right.$$
$$\left.\left. + \frac{\gamma_2}{C}RO - \rho_1 F\right)\}\right]$$

Since $(H - H^*)^2 = 0$ was the same as $H^2 - 2HH^* + H^{*2} = 0$.

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Implying $H(H - H^*) = H^*(H - H^*)$. **Thus** $H = H^*$. Similarly, $R = R^*$, $F = F^*$, $O = O^*$, $L = L^*$, $P = P^*$, $D = D^*$ and $C = C^*$ .Since $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_3 > 0$, $\sigma_4 > 0$, $\sigma_5 > 0$, $\sigma_6 > 0$ **and** $\sigma_7 > 0$, **were chosen such that** $\frac{dV}{dt} \leq 0$ and that the steady state E^* was globally asymptotically stable.

3.1.6 Bifurcation analysis

Lotka-Volterra system uses several numbers of parameters to describe and formulate the system. When the parameters in the model are altered, other types of dynamical behavior may change and the critical parameter values at which such changes happen are called bifurcation points. This study handled the transition factors of the prey-predator interactions. This adjustment with respect to small changes is called Hopf bifurcation. Hopf bifurcation occurred at the point where the system had non- hyperbolic equilibrium point and has purely imaginary eigen values.

This study considered ω_2, π_3 as the bifurcation parameters and ω_2^*, π_3^* as the turning point for values of the bifurcation parameters.

Theorem 3

Positive equilibrium over real domain R was considered. Let $\psi(0,\infty) \rightarrow R$ be the continuous differential function of ω_2. Then, $\psi(\omega_2) = M_1(\omega_2)M_2(\omega_2) - M_8(\omega_2)$. Since ω_2^* is the positive root of $\psi(\omega_2) = 0$.

Thus, Hopf bifurcation of inner equilibrium $E(H^*,R^*,F^*,O^*,L^*,P^*,D^*,C^*)$ occurred at $\omega_2 = \omega_2^*$ if and only if;

i) $\psi(\omega_2^*) = 0$

ii) $R_8(\omega_2^*)R_6(\omega_2^*) + R_7(\omega_2^*)R_5(\omega_2^*) + R_2(\omega_2^*)R_4(\omega_2^*) + R_1(\omega_2^*)R_3(\omega_2^*) \neq 0$

Proof:

Considering the first condition above that $\psi(\omega_2) = 0$, the characteristic equation of the variational matrix, $a_6\square^3 + a_7\square^2 + a_8\square + a_9 = 0$.

$\left(a_6\square^2 + a_8\right)(\square + a_1) = 0$

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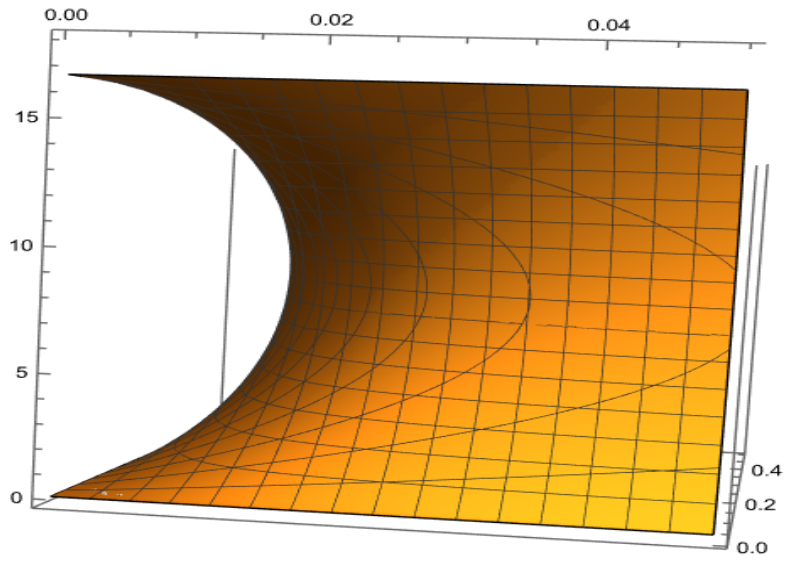
The roots of the equation are $\tau_1 = -a_1$, $\tau_2 = -i\sqrt{a_2}$, $\tau_3 = -i\sqrt{a_3}$

Therefore, it is asymptotically stable since all the roots are negative.

Backward bifurcation

A backward bifurcation occurs when a parameter is adjusted and a stable equilibrium point loses stability. However, the system does not create two new equilibrium points; rather, it returns to the region where the stable equilibrium point was initially stable. In our model this exhibited when we plot ω_2, π_3 and H using Mathematica software

We obtain



This exhibits that when the population increases up to 10 billion, where $\omega_2 = 0.2$ and $\pi_3 = 0.015$ the system starts going back to the region where it was stable originally.

3.2 Model simulation and data fitting

MATLAB2019a was utilized in numerical simulation to illustrate the non-linear ODE's dynamic behavior in the system (1)–(8). The simulations were run with the initial conditions and parameter values (taken from the literature review and graphically depicted) in Tables 1 and 2.

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From Fig. 1, when ω_3 was increased from 0.05234 to 0.9234 the CO_2 emissions increased by 3.75 percent. When α_2 increased from 0.1165 to 0.5165, the CO_2 emissions increased by 29.75 percent. When Φ_2 is increased from 0.5065 to 0.8065, the CO_2 emissions increased by 11.33 percent.

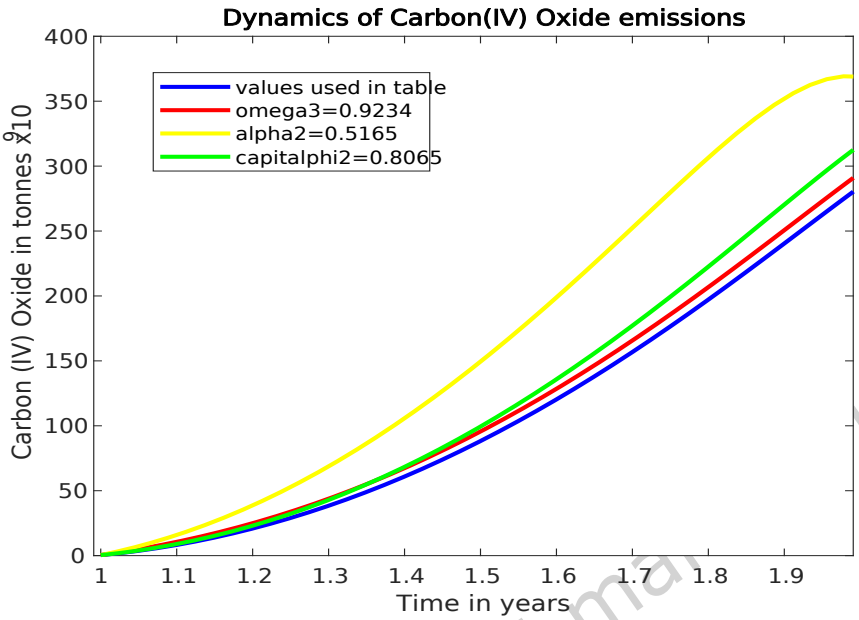

Fig. 1

From Fig. 2, when θ_2 was increased from 0.1711 to 0.4711, CO_2 emissions increased by 31.53 percent. When β_2 was increased from 0.1999 to 0.3999, the CO_2 emissions increased by 22.05 percent. When ε_2 was increased from 0.08611 to 0.28611, CO_2 emissions increased by 1.872 percent.

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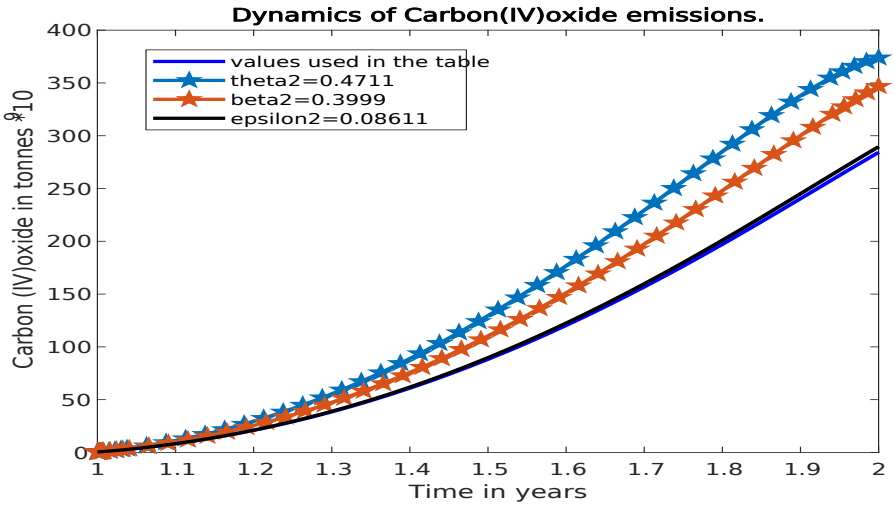

Fig. 2

From Fig. 3, when γ_2 was increased from 0.05914 to 0.25914, CO_2 emissions increased by 5.598 percent. When δ_2 was increased from 0.2494 to 0.5494, CO_2 emissions increased by 1.241 percent. When ρ_2 was increased from 0.1109 to 0.7109, the CO_2 emissions decreased by 25.37 percent.

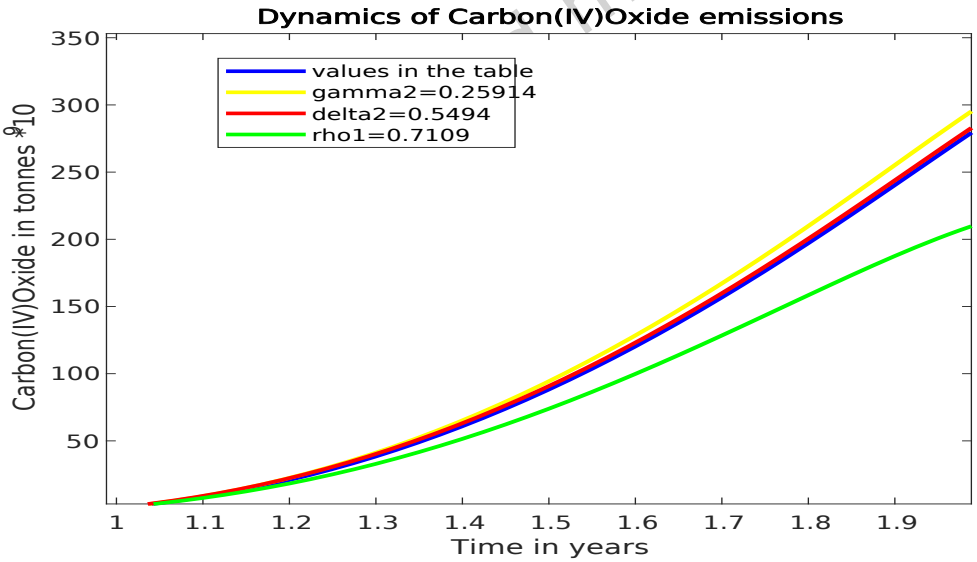

Fig. 3

4 Least square method and Newton's law of cooling

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A well-liked mathematical technique for data fitting, regression analysis, and predictive modeling is the least squares method. By minimizing the sum of squared discrepancies between the observed data points and the anticipated values, it assists in identifying the best-fit line or curve. It is a mathematical process that selects (estimates) a mathematical model's unknown parameters so that the sum of squares of a selected mistake is as little as possible.

Finding the best-fitting curve or line of best fit for a collection of data points is accomplished by minimizing the sum of squares of the offsets (residual portion) of the points from the curve using the least squares approach. The trend of results is quantitatively calculated throughout the process of determining the relationship between two variables. Regression analysis is the name given to this procedure. One technique for regression analysis is curve fitting. The least squares approach to equation fitting approximates the curves to the provided raw data.

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It should be understood that curves fitted to a given data set are not always unique. Finding a curve that deviates as little as possible from all of the recorded data points is therefore necessary. The least-squares approach is used to determine this, which is referred to as the best-fitting curve.

Newton’s law of cooling describes the rate at which an exposed body changes temperature through radiation. Newton's law of cooling, a scientific principle in the study of heat transmission, asserts that the rate at which a body loses heat is directly proportional to the temperature differential between the body and its surroundings. The requirement that the temperature differential be minimal and the nature of the heat transfer mechanism be constant is commonly added to the rule as a qualification.

As long as the temperature differential between the item and its surroundings is minimal, this rate is roughly proportional to the difference. In mathematical form, it may be written as

$\frac{dT(t)}{dt} = \lambda(T(t) - Ta)$,where $T(t)$ is the temperature of the body at any time t, and T_a is the ambient temperature (Mondol et al. 2018).

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The following secondary data for CO_2 emissions and the corresponding annual warming was obtained as (Tiseo 2023).

Year	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023
T	0	1	2	3	4	5	6	7	8	9
CO2 in billion metric tons	35.47	35.46	35.46	36.03	36.77	37.04	35.01	36.82	37.15	37.55
Annual warming in ℃	1.09	1.23	1.37	1.27	1.20	1.33	1.36	1.20	1.25	1.54

Using Microsoft Excel to fit data to model using Newton's law of cooling in the form $\frac{dT(t)}{dt} = kC(t)$, we obtain the following;

Year	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023		
T	0	1	2	3	4	5	6	7	8	9		
Carbon(iv)	35.47	35.46	35.46	36.03	36.77	37.04	35.01	36.82	37.15	37.55		
k	0.0016	0.003407717	0.00322381	0.00141	0.00065	0.00107	0.00105	0.00037	0.00046	0.00102	Max K	0.0034077
											Min K	0.0003730
Time	0	1	2	3	4	5	6	7	8	9	Average	0.0014288
Year	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023		
Annual wa	1.09	1.23	1.37	1.27	1.2	1.33	1.36	1.2	1.25	1.54	Total Error Sum	3.94114E-1
Model	1.09	1.23000142	1.36999942	1.27	1.2	1.33	1.36	1.2	1.25	1.54		
Error Sum	0	2.01776E-12	3.3459E-13	2.9E-13	5.2E-13	8.6E-15	1.1E-13	4.7E-15	4.3E-13	2.3E-13	3.94114E-12	

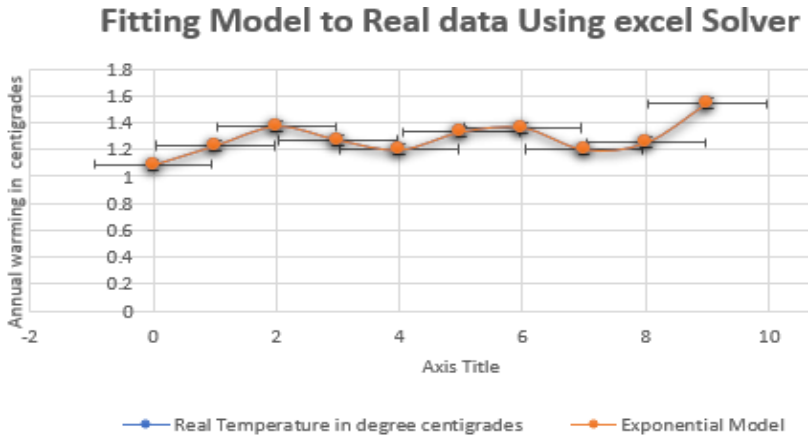

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Fig. 4

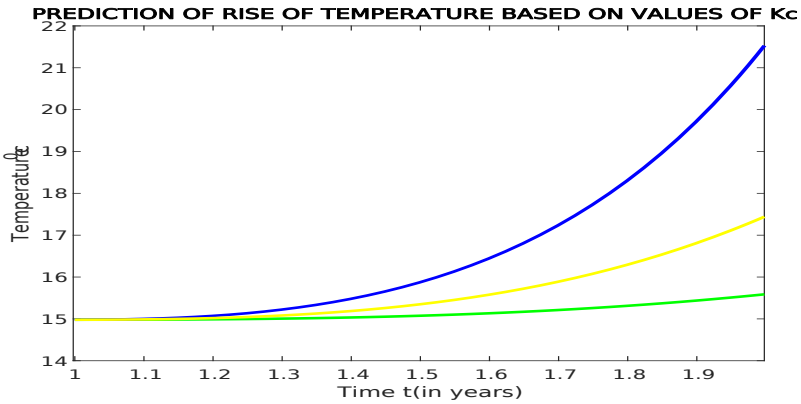

Fig. 5

From the simulations using the estimated maximum value of k_c = 0.003407717 the temperature will rise by 6.5748^0c which seems unrealistic, estimated minimum value of k_c = 0.000373026 the temperature will rise by 2.456^0c within the next two years and estimated average value of k_c = 0.001428872 the temperature will rise by 0.5903^0c within two years.

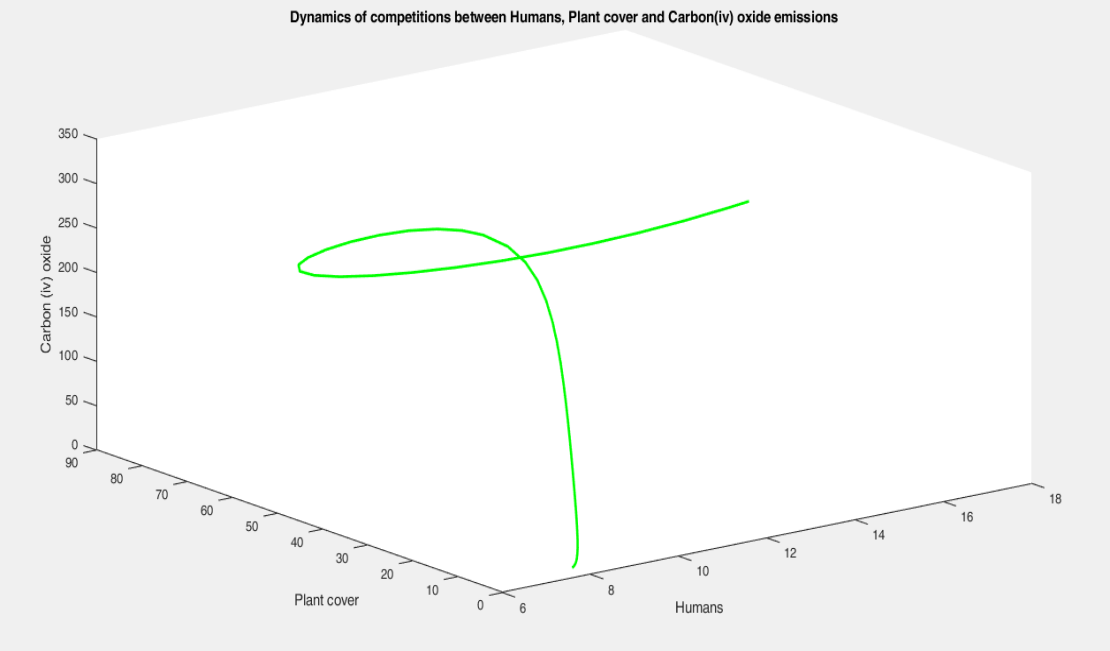

Fig. 6

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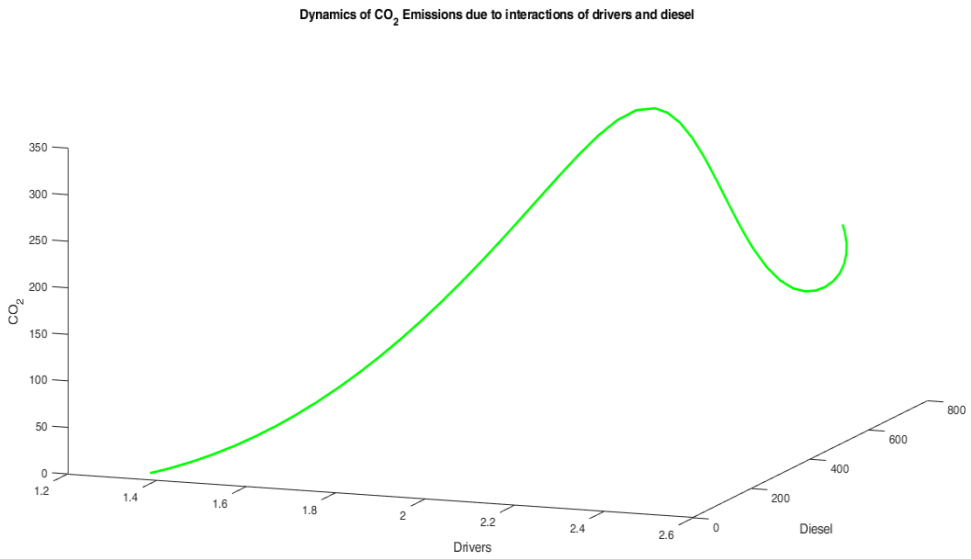

Fig. 7

It is evident from the above interactions that if the humans increase from 8 billion to 17.9227, the tree cover from 4.06 billion ha to 61.8461, the emissions will increase from 0.42186 parts per billion to 207.927. This clearly shows humans are the key contributors to the emissions through their activities. When drivers increase from 1.36 billion to 2.1546, the diesel used will rise from 37.3395 to 484.5954 the emissions will increase from 0.42186 parts per billion to 346.8193. This clearly shows humans are the key contributors to the emissions through their activities.

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5 Conclusions

This work aimed modeling the Role of Fossil Fuels and Human Activities in Carbon (IV) Oxide Emissions and Global Warming. A system of eight first -order non -linear ordinary differential equations that explain the dynamics of emissions of CO_2 helped to achieve this goal. In this work, method of characteristic polynomial, positivity and boundedness of the solution, equilibrium points were used to examine the behavior of the deterministic model. The Routh -Hurwitz criteria for stability were used to determine the local stability while the Castillo -Chavez approach was used to determine the global stability. Bifurcation analysis was also carried out. The model suggests that CO_2 emissions resulting from the

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interactions of fossil fuels and the human activities is the major cause of global warming. The model however did not address all the products that are obtained during the refinery of crude oil, other sources of CO_2 and other gases that also have significant contributions to the global warming. The model having considered the global level could not account for advection and diffusion of CO_2 which may vary from one region to another. Nevertheless, the policy makers can advocate other sources of energy that will not have the adverse impacts on climate thus making the world a better place by averting the worrying trend of rising temperature.

6 Recommendations

Further studies should focus on emission of other gases emitted such as methane, nitrous (I) oxide and Fluorinated gases so that as the CO_2 emissions is tackled other gases are also addressed so as achieve the Sustainable Development Goal number 13 on Climate Action that targets the reduction of greenhouse Gases emissions.

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Statements and declarations

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Competing interests

Authors declare no competing interests whatsoever.

Data availability

Data used was obtained from the literature.

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