International Journal of Statistics and Applied Mathematics



ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2026; 11(2): 103-107
© 2026 Stats & Maths
https://www.mathsjournal.com
Received: 16-12-2025
Accepted: 18-01-2026

Julia Wanjiku Karunditu
Department of Pure and Applied Sciences, Kirinyaga University, Kerugoya, Kenya

Cyrus Gitonga Ngari
Department of Pure and Applied Sciences, Kirinyaga University, Kerugoya, Kenya

Peter Njori Wanjohi
Department of Pure and Applied Sciences, Kirinyaga University, Kerugoya, Kenya

Corresponding Author:
Julia Wanjiku Karunditu,
Department of Pure and Applied Sciences, Kirinyaga University, Kerugoya, Kenya

A caputo–Fabrizio fractional model for Escherichia coli dynamics in wastewater treatment systems

Julia Wanjiku Karunditu, Cyrus Gitonga Ngari and Peter Njori Wanjohi

DOI: https://www.doi.org/10.22271/maths.2026.v11.i2b.2264

Abstract
Using bacteriophage predation, bacterial resistance, and Monod-type growth kinetics, this study creates and evaluates a multi-stage mathematical model for the dynamics of Escherichia coli (E. coli) in wastewater treatment systems. The pre-primary, primary, secondary, and tertiary stages of wastewater treatment are represented by a seven-compartment deterministic model. Memory and hereditary effects in bacterial dynamics are captured by extending the traditional integer-order model to a Caputo–Fabrizio fractional-order framework, which includes both constant and variable fractional orders. Fixed-point theory is used to establish the existence and uniqueness of solutions. The three-step Adams-Bashforth approach is used to do numerical simulations that compare formulations of integer order, constant fractional order, and variable fractional order.
Results indicate that the variable fractional-order model provides enhanced flexibility and improved representation of transient dynamics relative to classical approaches. The proposed framework offers a mathematically rigorous and adaptable tool for analysing *E. coli* control strategies in wastewater treatment, with potential relevance to public health and Sustainable Development Goals related to water and sanitation.

Keywords: Caputo-Fabrizio derivative, E. Coli, wastewater treatment, fractional-order model, bacteriophage predation, Monod growth, and resistance

1. Introduction
Escherichia coli (E. coli) contamination of wastewater has been a serious global environmental and public health concern. Pathogenic strains of Escherichia coli found in both treated and untreated sewage represent a threat to both industrialized and developing nations and help in the spread of waterborne illnesses. Therefore, attaining Sustainable Development Goal 6 which calls for universal access to clean water and sanitation and safeguarding public health depend on effective wastewater treatment(Ghosh, I., Chowdhury, M. S. H., Aznam, S. M., & Rashid, 2021) [4].
In order to comprehend the movement, permanence, and management of microbiological pollutants in wastewater systems, mathematical modeling has emerged as a crucial tool. Models of classical integer-order differential equations have been used extensively to explain the dynamics of contaminants and their treatment procedures(Ahmed *et al*., 2023) [1]. The memory, latency, and genetic consequences that result from biological and environmental processes such as bacterial adaptability, resistance development, and treatment history which may not be sufficiently captured by such models.
The ability to handle derivatives of non-integer order in fractional calculus makes it a logical extension of classical modeling frameworks. Complex dynamical systems with memory effects can be more accurately and flexibly represented using fractional-order models (Dokuyucu & Dutta, 2020) [2]. Because it uses a nonsingular exponential kernel, the Caputo–Fabrizio fractional derivative, in particular, avoids the singularity that occurs in the traditional Caputo derivative and has been effectively used in several biological and environmental applications (Ahmed *et al*., 2023) [1].
For E. coli dynamics, we create a multi-stage wastewater treatment model in this work that

takes into account resistance mechanisms, bacteriophage predation, and Monod-type bacterial growth. Both integer-order and Caputo-Fabrizio fractional-order forms, along with a variable fractional-order extension, are used to formulate the model.

The primary goals of this work are to

1. Develop a multi-compartment model for the dynamics of bacteriophages and E. coli in wastewater treatment systems that is motivated by biology.
2. Expand the model to a fractional-order Caputo-Fabrizio framework.
3. Prove the solutions' existence and uniqueness.
4. Use numerical simulations to compare integer-order, constant fractional-order, and variable fractional-order models.

This is how the rest of the paper is structured. Basic ideas of the Caputo-Fabrizio fractional operator are presented in Section 2. The model formulation is explained in Section 3. The existence and uniqueness of solutions are theoretically analyzed in Section 4. The simulation methodology and numerical framework are described in Section 5. Numerical results and implications are discussed in Section 6. Concluding thoughts are finally presented in Section 7.

2. Preliminaries on the Caputo–Fabrizio Fractional Operator

The definitions and characteristics of the Caputo-Fabrizio fractional derivative that are essential for the theoretical examination of the suggested model are briefly reviewed in this section. suppose $\mathcal{H}(c,d) = \{\vartheta: \vartheta \in L^2(c,d), \text{and } \vartheta' \in L^2(c,d)\}$, where (c,d) is the space of square integrable functions on (c,d) .

For $0 < \alpha < 1$, the Caputo-Fabrizio fractional derivative for $\vartheta \in \mathcal{H}'(c,d)$ is defined by

$$CF_{D_k^{\alpha}}\vartheta(K) = \frac{1}{1-\alpha}\int_c^k \vartheta'(y)\exp[-\alpha\frac{K-y}{1-\alpha}]dx, \text{ where k} \geq \text{c}$$

The corresponding Caputo Fabrizio fractional integral (Losada & Nieto, 2015) [5] is given by

$$CF_{I_t^{\alpha}}\vartheta(K) = (1-\alpha)\vartheta(K) + \alpha\int_c^k \vartheta(y)\,dy,$$

These operators are appropriate for modeling biological and environmental systems with fading memory because they enable the introduction of memory effects with a nonsingular exponential kernel(Losada & Nieto, 2015) [5].

3. Model Formulation

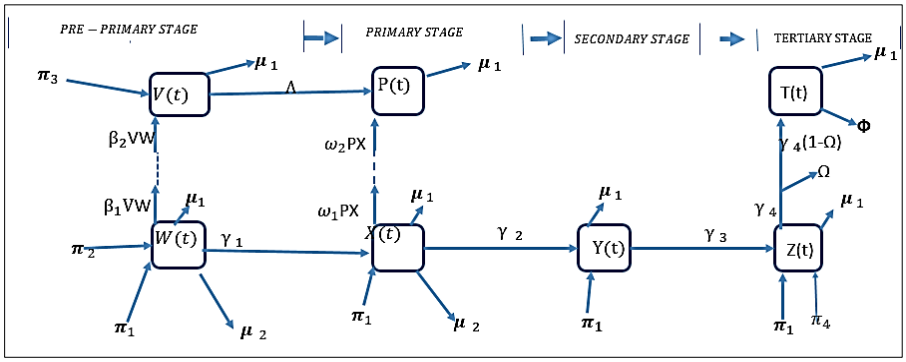
The wastewater treatment system is divided into multiple stages representing pre-primary, primary, secondary, and tertiary treatment. The model tracks concentrations of E. coli and bacteriophages across these stages, as well as concentrations of a resistant *E. coli* class.

3.1 State Variables

Let's

- **W(t):** Represents E. coli concentration in the pre-primary stage,
- **X(t):** Represents the concentration of E. coli in the initial stage,
- **Y(t):** Represents the secondary stage concentration of E. coli
- **Z(t)):** Represents E. coli concentration in the tertiary stage,
- **T(t):** Represents the tertiary stage concentration of resistant E. coli,
- **V(t):** Represents the concentration of bacteriophages in the pre-primary stage
- **P(t):** Represents the bacteriophage concentration in the primary stage.

3.3 Model Flowchart



3.4 Integer-Order Model

The integer-order model is given by the system of ordinary differential equations (1-7):

$$\frac{dW}{dt} = (\pi_1 + \pi_2 - \mu_1 W - \mu_2 W - \beta_1 WV + \frac{\pi_m WS}{K_2 + S} - \gamma_1 W)\left(1 - \frac{N}{A}\right) \quad (1)$$

$$\frac{dX}{dt} = \left(\gamma_1 W + \pi_1 - \mu_1 X - \mu_2 X - \omega_1 XP + \frac{\pi_m XS}{K_2 + S} - \gamma_2 X\right)\left(1 - \frac{N}{A}\right) \quad (2)$$

$$\frac{dY}{dt} = (\gamma_2 X + \pi_1 - \mu_1 Y + \frac{\pi_m YS}{K_2 + S} - \gamma_3 Y)\left(1 - \frac{N}{A}\right) \quad (3) \quad \frac{dZ}{dt} = (\gamma_3 Y + \pi_1 + \pi_4 - \mu_1 Z + \frac{\pi_m ZS}{K_2 + S} - \gamma_4 Z)\left(1 - \frac{N}{A}\right) \quad (4)$$

$$\frac{dT}{dt} = (\gamma_4 Z(1 - \Omega) - \mu_1 T + \frac{\pi_m TS}{K2 + S} - \Phi T)\left(1 - \frac{N}{A}\right) \quad (5)$$

$$\frac{dV}{dt} = \left(\beta_2 WV + \pi_3 - \mu_1 V + \frac{\pi_m VS}{K_2 + S} - \Lambda V\right)\left(1 - \frac{B}{A}\right) \quad (6)$$

$$\frac{dP}{dt} = \left(\Lambda V + \omega_2 XP - \mu_1 P + \frac{\pi_m PS}{K_2 + S}\right)\left(1 - \frac{B}{A}\right) \quad (7)$$

3.5 Assumptions of the Model

The following presumptions form the basis of the model:

1. It is possible to measure the concentrations of bacteriophages and E. coli per volume.
2. Bacteriophage predation, competition (such as algae), and treatment procedures all have an impact on E. coli populations.
3. The growth of bacteria is described by Monod-type Kinetics.
4. Only the pre-primary and main stages are important for bacteriophages.
5. The effects of evaporation, diffusion, and adsorption are minimal.
6. The tertiary stage is where resistance may emerge.

3.6 Fractional-Order Model

A Caputo-Fabrizio fractional-order model is created by extending the integer-order system as

$$CF_{D^\alpha}\mathrm{W(t)} = F_i\ (t, W, X, Y, Z, T, V, P)$$

$$CF_{D^\alpha}\mathrm{X(t)} = F_i\ (t, W, X, Y, Z, T, V, P)$$

$$CF_{D^\alpha}\mathrm{X(t)} = F_i\ (t, W, X, Y, Z, T, V, P)$$

where the right-hand sides of the corresponding integer-order equations are represented by the functions F_i and $0 < \alpha < 10 < 10 < \alpha < 1$(Dokuyucu & Dutta, 2020) [2].

4. Solutions Existence and Uniqueness

Using fixed-point theory, we establish the existence and uniqueness of solutions for the Caputo–Fabrizio fractional-order model. Under acceptable boundedness and Lipschitz continuity assumptions on the nonlinear kernels F_i, the corresponding integral equations can be proven to admit a unique solution on a finite time period (Edessa, 2022) [3].

For the sake of conciseness, the Supplementary Material contains comprehensive proofs based on contraction mappings and Lipschitz conditions.

5. The Numerical System

A three-step Adams–Bashforth method adapted to the Caputo–Fabrizio fractional derivative is used to obtain numerical solution (Edessa, 2022) [3] s. Simulations are performed for constant fractional orders (e.g., $\alpha = 0.7,0.8,0.9$ variable fractional-order cases, and integer-order cases $\alpha = 1$.

Time step sizes $h = 0.01, 0.05,$ and 0.1 are used to assess numerical stability and convergence.

6. Results and Discussion

Figure 1

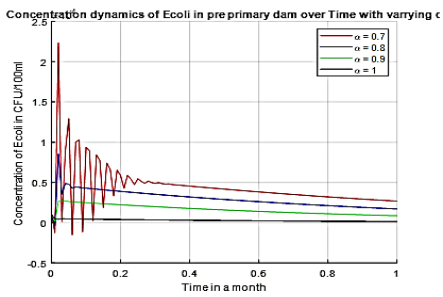


Figure 1 data statistics

Alpha (α)	Minimum		Maximum		Range	
	Time(in a month)	CFU/100ml	Time(in a month)	CFU/100ml	Time(in a month)	CFU/100ml
0.7	0	-1513	1	2.234e+04	1	2.385e+04
0.8	0	-396.6	1	8570	1	8967
0.9	0	146.6	1	2718	1	2571
1.0	0	132.3	1	1000	1	867.7

Figure 2

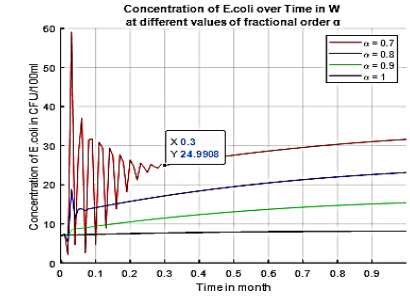


Figure 2 data statistics

Alpha (α)	Minimum		Maximum		Range	
	Time(in a month)	CFU/100ml	Time(in a month)	CFU/100ml	Time(in a month)	CFU/100ml
0.7	0	2.049	1	59.07	1	57.07
0.8	0	5.227	1	23.13	1	17.9
0.9	0	6.605	1	15.4	1	8.799
1.0	0	7.011	1	8.109	1	1.098

Figure 1 shows that the fractional order(α) has a significant stabilizing effect on *E. coli* concentration dynamics in the pre-primary stage.
For lower fractional orders (α=0.7), the system exhibits

- Large-amplitude oscillations.
- Very high maximum values (up to 2.234×1042.234 \times 10^42.234×104 CFU/100ml).
- Extremely large ranges (up to 2.385×1042.385 \times 10^42.385×104 CFU/100ml).

As α→1to 1α→1 (classical integer-order case)

- Oscillations are strongly damped.
- Peak concentrations are significantly reduced (maximum ≈ 1000 CFU/100ml). The range decreases to only 867.7 CFU/100ml

Figure 2: Effects of Long-Term Growth and Memory in W (Figure 2)
The fractional order has a significant impact on the long-term buildup of E. coli in compartment W, as shown in Figure 2.
Important Findings: • A lower α results in:
Long-term concentrations are higher.
i) Larger maximum values (59.07 CFU/100ml for α=0.7). • Higher α (approaching 1): o Produces smaller maximum (8.109 CFU/100ml for α=1).
ii) Significantly reduced ranges (only 1.098 CFU/100ml for α=1).
This means that increasing fractional order promotes effective damping and reduces bacterial survival.

Figure 7

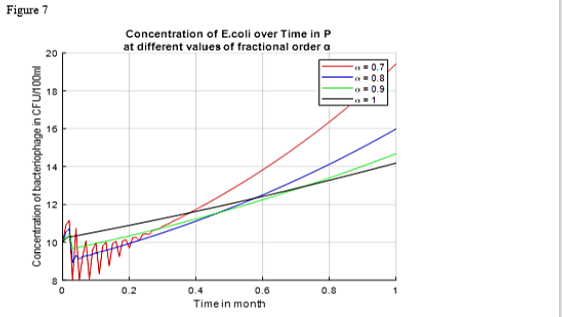
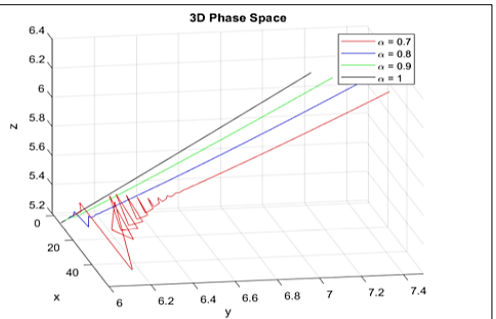


Figure 7 data statistics

Alpha (α)	Minimum		Maximum		Range	
	Time(in a month)	CFU/100ml	Time(in a month)	CFU/100ml	Time(in a month)	CFU/100ml
0.7	0	8.01	1	19.41	1	11.4
0.8	0	8.93	1	15.98	1	7.054
0.9	0	9.669	1	14.68	1	5.007
1.0	0	10	1	14.18	1	4.182



The relevance of memory effects in bacterial dynamics is demonstrated by Figure 7 and the accompanying 3D phase pictures, which show that decreasing the fractional order α increases E. coli persistence and intensifies oscillatory behavior. While lower α values introduce transient instability and higher peak concentrations, higher α values promote smoother trajectories and faster convergence to equilibrium. These findings demonstrate that fractional-order modelling reflects delayed system reactions inherent in wastewater treatment procedures more realistically than standard integer-order formulas.
Across all compartments, the fractional order α consistently governs the balance between memory-induced amplification and dynamical damping. Lower fractional orders boost the endurance of both sensitive and resistant E. coli populations, amplify

transient oscillations, and raise peak bacterial and phage concentrations. On the other hand, the system moves into a more stable regime with less variability, the removal of nonphysical oscillations, and enhanced numerical robustness as $\alpha \to 1$. These results reveal that fractional-order memory has a major role in early-stage and predator–prey dynamics, while advanced treatment stages remain comparatively insensitive, demonstrating substantial intrinsic stability of tertiary treatment mechanisms.

Compared to the integer-order model, fractional-order models exhibit stronger memory effects and a larger decrease, in line with numerical results. The variable fractional-order formulation offers greater flexibility, allowing the system to adjust to time-varying dynamics associated with treatment efficiency and biological responses (Dokuyucu & Dutta, 2020) [2].

Notably, delayed reactions and transitory behavior in E. coli concentrations are better captured by the variable-order model. These results indicate that variable fractional-order models may offer higher forecasting capability for complex wastewater treatment dynamics.

Global Mathematical Conclusion

The numerical results demonstrate that the fractional order α plays a critical regulatory role in the transient and long-term dynamics of *E. coli* concentrations. Lower fractional orders induce strong memory effects that amplify oscillations, increase peak concentrations, and enhance persistence across treatment stages. As $\alpha \to 1$, the system exhibits increased damping, reduced transient overshoot, and improved stability. Notably, fractional-order effects are most pronounced in the early treatment stages, while the tertiary stage remains relatively insensitive to memory, indicating strong intrinsic stability of advanced treatment processes.

7. Conclusion

This study used a Caputo–Fabrizio fractional-order framework to present a multi-stage wastewater treatment model for E. coli dynamics that included bacteriophage predation, resistance, and Monod-type growth. While numerical simulations demonstrated the benefits of both constant- and variable-order fractional formulations over traditional integer-order models, theoretical analysis confirmed the existence and uniqueness of solutions.

The proposed model provides a versatile and mathematically rigorous tool for studying microbial dynamics in wastewater treatment systems.

Future research will focus on parameter estimates using actual wastewater data and on additional model validation to provide useful decision support.

References

1. Ahmed I, Akgül A, Jarad F, Kumam P, Nonlaopon K. A Caputo-Fabrizio fractional-order cholera model and its sensitivity analysis. Mathematical Modelling and Numerical Simulation with Applications. 2023;3(2):170–187. doi:10.53391/mmnsa.1293162.
2. Dokuyucu MA, Dutta H. A fractional order model for Ebola virus with the new Caputo fractional derivative without singular kernel. Chaos, Solitons and Fractals. 2020;134:109717–109717. doi:10.1016/j.chaos.2020.109717.
3. Edessa GK. Existence and uniqueness solution of the model of enzyme kinetics in the sense of Caputo-Fabrizio fractional derivative. International Journal of Differential Equations. 2022;2022:1345919–1345919. doi:10.1155/2022/1345919.
4. Ghosh I, Chowdhury MSH, Aznam SM, Rashid MM. Measuring the pollutants in a system of three interconnecting lakes by the semianalytical method. Journal of Applied Mathematics. 2021;2021:6664307–6664307. doi:10.1155/2021/6664307.
5. Losada J, Nieto JJ. Properties of a new fractional derivative without singular kernel. Progress in Fractional Differentiation and Applications. 2015;1(2):87–92. doi:10.12785/pfda/010202.
6. Belew B, Melese D. Modeling and analysis of predator-prey model with fear effect in prey and hunting cooperation among predators and harvesting. Journal of Applied Mathematics. 2022;2022:2776698–2776698. doi:10.1155/2022/2776698.
7. Fauzi SHM, Jamil NM. Wastewater treatment process: A modified mathematical model for oxidation ponds. Journal of Advanced Research in Fluid Mechanics and Thermal Sciences. 2021;86(1):76–86. doi:10.37934/arfmts.86.1.7686.
8. Gutierrez A, Jain S, Bhargava P, Hamblin M, Lobritz MA, Collins JJ. Understanding and sensitizing density-dependent persistence to quinolone antibiotics. Molecular Cell. 2017;68(6):1147–1154.e3. doi:10.1016/j.molcel.2017.11.012.
9. Hernández-Crespo C, Fernández-Gonzalvo MI, Miglio RM, Martín M. *Escherichia coli* removal in a treatment wetland–pond system: A mathematical modelling experience. Science of the Total Environment. 2022;839:156237–156237. doi:10.1016/j.scitotenv.2022.156237.
10. Perulli GD, Gaggia F, Manfrini L, Di Gioia D, Toscano A, Morandi B. The fate of bacteria in urban wastewater-irrigated peach tree: A seasonal evaluation from soil to canopy. Scientia Horticulturae. 2024;323:112520–112520. doi:10.1016/j.scienta.2023.112520.
11. Prakasha DG, Veeresha P. Analysis of the lakes pollution model with Mittag-Leffler kernel. Journal of Ocean Engineering and Science. 2020;5(4):310–322. doi:10.1016/j.joes.2020.01.004.
12. Rodriguez-Perez S, Cabeza Sanchez A, Lopez-Abelairas M. Modelling and mathematical optimisation of wastewater treatment in food industries. Open Research Europe. 2022;2:1–20. doi:10.12688/openreseurope.14737.1.
13. Savini F, Luca V, Bocedi A, Massoud R, Park Y, Mangoni ML, Stella L. Cell-density dependence of host-defense peptide activity and selectivity in the presence of host cells. ACS Chemical Biology. 2017;12(1):52–56. doi:10.1021/acschembio.6b00910.