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CYCLE INDEX FORMULAS FOR DIHEDRAL GROUP (D_n) ACTING ON ORDERED TRIPLES

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ABSTRACT

The cycle index of dihedral group D_n acting on the set X of the vertices of a regular n-gon was studied by Harary and Palmer in 1973 (See [1]). In this paper we study the cycle index formulas of D_n acting on unordered triples from the set $X = \{1,2, \dots, n\}$. In each case the actions of the cyclic part and the reflection part are studied separately for both an even value of n and an odd value of n.

1. Introduction

The concept of the cycle index was discovered by Polya (See [2]) and he gave it its present name. He used the cycle index to count graphs and chemical compounds via the Polya's Enumeration Theorem. More current cycle index formulas include the cycle index of the reduced ordered triples groups $S_n^{[3]}$ (See [3]) which was further extended by Kamuti and Njuguna to cycle index of the reduced ordered r-group $S_n^{[r]}$ (See [4]). The Cycle Index of Internal Direct Product Groups was done in 2012 (See [5]).

2. Definitions and Preliminaries

Definition 1.
A cycle index is a polynomial in several variables which is structured in such a way that information about how a group of permutations acts on a set can be simply read off from the coefficients and exponents.
Definition 2.
A cycle type of a permutation is the data of how many cycles of each length are present in the cycle decomposition of the permutation.

Definition 3.
A monomial is a product of powers of variables with nonnegative integer exponents possibly with repetitions.

Preliminary result 1

To investigate the cycle index of ordered triples, we first highlight some results for details and proof (See [3]).
Suppose $g \epsilon D_n$ with $mon(g) = t_1^{\alpha_1}, t_2^{\alpha_2} \dots t_n^{\alpha_n}$ and g' is the corresponding permutation in $D_n^{[3]}$. To obtain an expression for $Z\left(D_n^{[3]}\right)$ we need to find $mon(g')$ for every $g \epsilon D_n$.
We have to consider the following separate contributions from g to the corresponding term of $mon(g')$.

i. Contributions whereby all the three points lying in a triple come from a common cycle of g of length $m \geq 3$ in g.

In this case we have; $t_m^{\alpha_m} \longrightarrow b_m^{(m-1)(m-2)\alpha_m}$ ————(2.1)

ii. Contributions whereby a pair of points in a triple comes from a common cycle of length r and the remaining point comes from a different cycle of length s and if there are α_r cycles of length r and α_s cycles of length s then the contributions will be;

if $r \neq s$ $\quad t_r^{\alpha_r} t_s^{\alpha_s} \longrightarrow b_{[r,s]}^{3(r-1)(r,s)\alpha_r\alpha_s}$ ———— (2.2)

if $r = s = m$ $\quad t_m^{\alpha_m} \longrightarrow b_m^{6m(m-1)\binom{\alpha_m}{2}}$ ———— (2.3)

iii. Contributions whereby all the three points of a triple come from different cycles of g (say d, e and f) and there are α_d cycles of length d, α_e cycles of length e and α_f cycles of length f then the contributions will be;

If $d \neq e \neq f$ $\quad t_d^{\alpha_d} t_e^{\alpha_e} t_f^{\alpha_f} \longrightarrow b_{[d,e,f]}^{\frac{3!def}{[d,e,f]}\alpha_d\alpha_e\alpha_f}$ ———— (2.4)

If $d = e \neq f$ $\quad t_d^{\alpha_d} t_f^{\alpha_f} \longrightarrow b_{[d,f]}^{3!d(d,f)\binom{\alpha_d}{2}\alpha_f}$ ———— (2.5)

If $d = e = f = m$ $\quad t_m^{\alpha_m} \longrightarrow b_m^{3!m^2\binom{\alpha_m}{3}}$ ———— (2.6)

Preliminary result 2

The cycle index formulas of dihedral group D_n acting on the set X of the vertices of a regular n-gon are given by:

$$Z_{Dn,X} = \frac{1}{2n}\left[\sum_{d|n} \phi(d) t_d^{\frac{n}{d}} + \frac{n}{2} t_1^2 t_2^{\frac{n-2}{2}} + \frac{n}{2} t_2^{\frac{n}{2}}\right] \qquad 2.7(a)$$

if n is even and

$$Z_{Dn,X} = \frac{1}{2n}\left[\sum_{d|n} \phi(d) t_d^{\frac{n}{d}} + n t_1 t_2^{\frac{n-1}{2}}\right] \qquad 2.7(b)$$

if n is odd. Where ϕ is the Euler's phi formula.
The proof to these important results can be found in several books and articles (e.g See[6], [7] and [1])

3. Cycle index of D_n acting on ordered triples.

We first consider the cyclic part.

We note that from 2.7(a) and 2.7(b) the cycle index of the cyclic part of D_n acting on the set X of the n vertices of a regular n-gon for both n even and odd is given by;

$$Z_{C_n} = \frac{1}{n}\sum_{d|n} \phi(d) t_d^{\frac{n}{d}}.$$

We now consider all the possible combinations that can be made from $t_d^{\frac{n}{d}}$.
If all the three elements in a triple come from a common cycle, then from (2.1) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{\frac{n}{d}(d-1)(d-2)} = b_d^{\frac{nd^2-3nd+2n}{d}} \qquad (3.1).$$

If a pair comes from a common cycle and the other element comes from a different cycle, then the two cycles must be of length d and hence from (2.3) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{6d(d-1)\binom{n/d}{2}} = b_d^{\frac{3dn^2-3nd^2-3n^2+3nd}{d}} \qquad (3.2).$$

If each of the three elements comes from its own cycle, then all the three cycles must be if length d and hence from (2.6) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{3!d^2\binom{n/d}{3}} = b_d^{\frac{n^3-3n^2d+2nd^2}{d}} \qquad (3.3).$$

Combining (3.1), (3.2) and (3.3) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{\frac{n^3-3n^2d+2nd^2}{d}+\frac{3dn^2-3nd^2-3n^2+3nd}{d}+\frac{nd^2-3nd+2n}{d}} = b_d^{\frac{n(n-1)(n-2)}{d}} \qquad (3.4).$$

Therefore the cycle index of C_n acting on ordered triples is given by;

$$Z_{C_n,X^{[3]}} = \frac{1}{n}\left(\sum_{d|n} \phi(d)\, b_d^{\frac{n(n-1)(n-2)}{d}}\right) \qquad (3.5).$$

3.1 If n is even

We now consider the reflection part $t_2^{\frac{n}{2}}$

If two elements in a triple comes from a common cycle of length two and the other element comes from a different cycle of length two then from (2.3), we have;

$$t_2^{\frac{n}{2}} \longrightarrow b_2^{6(2)(1)\binom{n/2}{2}} = b_2^{\frac{3n^2-6n}{2}} \quad \text{———} \quad (3.1.1).$$

If each of the elements in a triple comes from a different cycle of length two then from (2.6), we have;

$$t_2^{\frac{n}{2}} \longrightarrow b_2^{3!2^2\binom{n/2}{3}} = b_2^{\frac{n^2-6n^2+8n}{2}} \quad \text{———} \quad (3.1.2).$$

Combining (3.1.1) and (3.1.2), we have;

$$t_2^{\frac{n}{2}} \longrightarrow b_2^{\frac{n^2-6n^2+8n}{2}+\frac{3n^2-6n}{2}} = b_2^{\frac{n(n-1)(n-2)}{2}}.$$

But from 2.7(a) we have $\frac{n}{2}$ monomials of the form $t_2^{\frac{n}{2}}$ and hence $\frac{n}{2}$ monomials will be induced giving;

$$\frac{n}{2}t_2^{\frac{n}{2}} \longrightarrow \frac{n}{2}b_2^{\frac{n(n-1)(n-2)}{2}} \quad \text{———} \quad (3.1.3).$$

We now consider the part $t_1^2t_2^{\frac{n-2}{2}}$.

If a pair comes from a cycle of length two and the other element from a cycle of length two then from (2.3) we have;

$$t_2^{\frac{n-2}{2}} \longrightarrow b_2^{6(2)\binom{\frac{n-2}{2}}{2}} = b_2^{\frac{3(n^2+8-6n)}{2}} \quad \text{———} \quad (3.1.4).$$

If a pair comes from a cycle of length two and the other element from a cycle of length one then from (2.2) we have;

$$t_1^2t_2^{\frac{n-2}{2}} \longrightarrow b_2^{3n-6} \quad \text{———} \quad (3.1.5).$$

If each of the elements in a triple comes from its own cycles with two elements from cycles of length one and one from a cycle of length two, then from (2.5) we have;

$$t_1^2t_2^{\frac{n-2}{2}} \longrightarrow b_2^{3n-6} \quad \text{———} \quad (3.1.6).$$

If each of the elements in a triple comes from its own cycles with two from cycles of length two and one from a cycle of length one, then from (2.5) we have;

$$t_1^2t_2^{\frac{n-2}{2}} \longrightarrow b_2^{3n^2+24-18n} \quad \text{———} \quad (3.1.7).$$

If each of the elements in a triple comes from its own cycles of length two, then from (2.5) we have;

$$t_2^{\frac{n-2}{2}} \longrightarrow b_2^{\frac{n^3-12n^2+44n-48}{2}} \quad \text{———} \quad (3.1.8).$$

Combining (3.1.4), (3.1.5), (3.1.6), (3.1.7) and (3.1.8) we have;

$$t_1^2t_2^{\frac{n-2}{2}} \longrightarrow b_2^{\frac{n(n-1)(n-2)}{2}}.$$

But from 2.7(a) we have $\frac{n}{2}$ monomials of the form $t_1^2t_2^{\frac{n-2}{2}}$ and hence $\frac{n}{2}$ monomials will be induced giving;

$$\frac{n}{2}\,t_1^2t_2^{\frac{n-2}{2}} \longrightarrow \frac{n}{2}b_2^{\frac{n(n-1)(n-2)}{2}} \quad \text{———} \quad (3.1.9)$$

To get the cycle index of D_n acting on $X^{[3]}$ when n is even we combine (3.5), (3.1.3) and (3.1.9) giving;

$$Z_{D_n,X^{[3]}} = \frac{1}{2n}\left[\sum_{d|n}\phi(d)\,b_d^{\frac{n(n-1)(n-2)}{d}} + \frac{n}{2}b_2^{\frac{n(n-1)(n-2)}{2}} + \frac{n}{2}b_2^{\frac{n(n-1)(n-2)}{2}}\right]$$

$$= \frac{1}{2n}\left[\sum_{d|n}\phi(d)\,b_d^{\frac{n(n-1)(n-2)}{d}} + nb_2^{\frac{n(n-1)(n-2)}{2}}\right] \quad \text{———} \quad (3.1.10).$$

3.2 If n is odd

We consider the part $t_1t_2^{\frac{n-1}{2}}$

If a pair of the elements in a triple comes from a common cycle of length two and the third element comes from a cycle of length two then from (2.3) we have;

$$t_2^{\frac{n-1}{2}} \longrightarrow b_2^{12\binom{\frac{n-1}{2}}{2}} = b_2^{\frac{3n^2-12n+9}{2}} \quad \text{———} \quad (3.2.1).$$

If a pair comes from a common cycle and the other element comes from a cycle of length one, then from (2.2) we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow b_2^{3\frac{n-1}{2}} = b_2^{\frac{3n-3}{2}} \quad \text{———} \quad (3.2.2).$$

If each element comes from its own cycle with two elements coming from cycles of length two and one from a cycle of length one, then from (2.5) we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow b_2^{3!(2)\binom{\frac{n-1}{2}}{2}} = b_2^{\frac{3n^2-12n+9}{2}} \quad \text{———} \quad (3.2.3).$$

If each of the elements in a triple come from a different cycle of length two, then from (2.6) we have;

$$t_2^{\frac{n-1}{2}} \longrightarrow b_2^{3!2^2\binom{\frac{n-1}{2}}{3}} = b_2^{\frac{n^3-9n^2+23n-15}{2}} \quad \text{———} \quad (3.2.4).$$

Combining (3.2.1), (3.2.2), (3.2.3) and (3.2.4) we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow b_2^{\frac{n^3-9n^2+23n-15}{2}+\frac{3n^2-12n+9}{2}+\frac{3n-3}{2}+\frac{3n^2-12n+9}{2}} = b_2^{\frac{n(n-1)(n-2)}{2}}.$$

But from 2.7(b) we have n monomials of the form $t_1 t_2^{\frac{n-1}{2}}$ and hence n monomials will be induced giving;

$$n t_1 t_2^{\frac{n-1}{2}} \longrightarrow n b_2^{\frac{n(n-1)(n-2)}{2}} \quad \text{———} \quad (3.2.5).$$

To get the cycle index of D_n acting on $X^{[3]}$ if n is odd we combine (3.5) and (3.2.5) giving;

$$Z_{D_n,X^{[3]}} = \frac{1}{2n}\left[\sum_{d|n} \phi(d)\, d_d^{\frac{n(n-1)(n-2)}{d}} + n b_2^{\frac{n(n-1)(n-2)}{2}}\right] \quad \text{———} \quad (3.2.6).$$

Example 1

Let $n = 7$, then the dihedral group D_7 of degree 7 acting on ordered triples of the set $X = \{1,2,3,4,5,6,7\}$. Then;

$|D_7| = 14$, $d = 1,7$, $\phi(1) = 1$ and $\phi(7) = 6$.

Hence from (3.2.6) we have;

$$Z_{D_7,X^{[3]}} = \frac{1}{14}\left[b_1^{210} + 6b_7^{30} + 7b_2^{105}\right].$$

Example 2

Let $n = 8$, then the dihedral group D_8 of order 8 acting on ordered triples of the set $X = \{1,2,3,4,5,6,7,8\}$. Then;

$|D_8| = 16$, $d = 1,2,4,8$, $\phi(1) = 1$, $\phi(2) = 1$, $\phi(4) = 2$ and $\phi(8) = 4$

Hence from (3.1.10) we have;

$$Z_{D_8,X^{[3]}} = \frac{1}{16}\left[b_1^{336} + 2b_4^{84} + 9b_2^{168} + 4\, b_8^{42}\right].$$

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