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A study of W_5 curvature tensor in LP-Sasakian manifold

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Abstract
The object of the paper is to study the geometrical properties of W_5 - Curvature tensors in LP-Sasakian Manifold and prove some results.

Keywords: W_5 -flat, symmetric, semi-symmetric and recurrent

1. Introduction

A manifold M_n is said to be LP-Sasakian manifold if it admits a (1,1) tensor field, a C$^\infty$ 1 -form A and a Lorentzian metric g which satisfy the following properties [1]

(1.1) $\eta(\xi) = -1$,
(1.2) $\phi^2 X = X + \eta(X)\,\xi$,
(1.3) $g(\phi X, \phi Y) = g(X,Y) + \eta(X)\,\eta(Y)$,
(1.4) $g(X,\xi) = \eta(X)$, $\nabla_X \xi = \phi X$,
(1.5) $(\nabla_X \phi)(Y) = g(X,Y)\,\xi + \eta(Y)\,\xi + \eta(Y)\,X + 2\,\eta(X)\,\eta(Y)\,\xi$.

Where
∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Given than M_n is a LP-Sasakian Manifold with the structure (ϕ, ξ, η, g), we can deduce the following [1]

(1.6) $g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\,\eta(Y)$,
(1.7) $R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X$,
(1.8) $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$,
(1.9) $R(\xi,X)\xi = X + \eta(X)\xi$,
(1.10) $Ric(X,\xi) = n - 1\,\eta(X)$,
(1.11) $Ric(\phi X, \phi Y) = Ric(X,Y) + (n-1)\eta(X)\eta(Y)$.

for any vector fields X, Y, Z.
Pokhariyal [4] defined W_5-Curvature tensor as

$$W_5(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}[g(X,Z)QY - Ric(X,Z)Y]$$

Or

(1.12) $$W_5'(X, Y, Z, \xi) = R'(X, Y, Z, \xi) + \frac{1}{n-1}[g(X,Z)Ric(Y,\xi) - g(Y,\xi)Ric(X,Z)]$$

We shall use the above results in the following sections.

2. W_5 - Curvature Tensor in LP- Sasakian Manifold.

Definition 2.1: An LP-Sasakian Manifold M_n is said to be W_5 -flat if $W_5(X,Y)Z = 0$

Theorem 2.1: A W_5 -flat LP-Sasakian manifold is a flat manifold.

Proof

From (1.12) if W_5' (X, Y, Z, ξ) = 0

Then

R' (X, Y, Z, ξ) = $\frac{1}{n-1}[g(Y,\xi)Ric(X,Z) - g(X,Z)Ric(Y,\xi)]$

applying Ric (X, Y) = (n-1) g (X, Y) in the above equation we have

R' (X, Y, Z, ξ) = $\frac{1}{n-1}[g(Y,\xi)(n-1)g(X,Z) - g(X,Z)(n-1)g(Y,\xi)]$

R' (X, Y, Z, ξ) = $[g(Y,\xi)g(X,Z) - g(X,Z)g(Y,\xi)]$

R' (X, Y, Z, ξ) = 0

Hence the theorem.

3. W_5 - Symmetric LP- Sasakian Manifold

Definition 3.1: An LP- Sasakian Manifold is called W_5 - symmetric if

$\nabla_u W_5$(X, Y, Z) = W_5'(X, Y, Z, U) = 0

Theorem 3.1: A W_5-symmetric and W_5 - flat LP-Sasakian manifold is a flat manifold.

Proof

From the symmetric property we have

∇_u W_5 (X, Y) Z = R (X, Y, W_5 (Z, U, V)) - W_5 (R(X, Y, Z),U,V) -

W_5 (Z, R(X, Y,U),V) - W_5 (Z,U,R(X, Y,V))= 0

Expanding the above expression

(3.1) R (X, Y, W_5 (Z, U, V) = R' (X, Y, W_5 (Z, U, V), ξ)

R' (X, Y, W_5 (Z, U, V), ξ) = g(X,ξ) g(Y, W_5 (Z, U, V)) - g(Y,ξ)g(X, W_5 (Z, U, V))
= $\eta(X)$ W_5' (Y,Z,U,V) - $\eta(Y)$ W_5' (X,Z,U,V)

(3.2) W_5 (R (X, Y, Z), U, V) = W_5' (R (X, Y, Z), U, V, ξ)

W_5' (R (X, Y, Z), U, V, ξ) = R'(R (X, Y Z), U, V, ξ) + $\frac{1}{n-1}$ [g(R(X, Y, Z),V)Ric(U,ξ)
- g (U, ξ) Ric(R (X, Y, Z), V)] on using Ric (X, Y) = (n -1) g(X, Y) we have

W_5' (R (X, Y, Z), U, V, ξ) = R'(R (X, Y Z), U, V, ξ) + $\frac{n-1}{n-1}$ [g(R(X, Y, Z), V)g(U,ξ)
- g (U, ξ)g(R (X, Y, Z), V)]

W_5' (R (X, Y, Z), U, V, ξ) = R'(R (X, Y Z), U, V, ξ)

R'(R (X, Y Z), U, V, ξ) = g(U, V)g(R (X, Y Z),ξ) - g(R(X, Y, Z),V)g(U,ξ)
= g(U, V)R'(X, Y, Z, ξ) - η(U)R'(X, Y, Z, V)

(3.3) W_5 (Z, R(X, Y, U), V) = W_5' (Z,R(X, Y,U), V,ξ)

W_5' (Z,R(X, Y,U), V,ξ) = R'(Z, R(X, Y, U), V, ξ) + $\frac{1}{n-1}$ [g(Z,V)Ric(R(X,Y,U), ξ) -
g(R(X, Y, U),ξ)Ric(Z,V)]

on using Ric (X, Y) = (n -1)g(X,Y) we have

W_5' (Z,R(X, Y,U), V,ξ) = R'(Z, R(X, Y, U), V, ξ) + $\frac{n-1}{n-1}$ [g(Z,V)g(R(X,Y,U), ξ) -
g(R(X, Y, U),ξ)g(Z,V)]

W_5' (Z,R(X, Y,U), V,ξ) = R'(Z, R(X, Y, U), V, ξ)

R'(Z, R(X, Y, U), V, ξ) = g(R(X, Y, U),V)g(Z,ξ) - g(Z,V)g(R(X, Y, U),ξ)
= R' (X, Y, U, V)η(Z) - g(Z,V) R'(X,Y,U,ξ)

(3.4) W_5(Z, U, R(X, Y,V)) = W_5' (Z,U,R(X,Y,V),ξ)

W_5' (Z,U,R(X,Y,V),ξ) = R'(Z,U,R(X,Y,V),ξ) + $\frac{1}{n-1}$ [g(Z, R(X, Y,V))Ric(U,ξ)
- g (U,ξ)Ric(Z, R(X, Y,V))]

on using Ric(X, Y) = (n -1)g(X,Y) we have

W_5' (Z,U,R(X,Y,V),ξ) = R'(Z,U,R(X,Y,V),ξ) + $\frac{n-1}{n-1}$ [g(Z, R(X, Y,V))g(U,ξ)
- g (U,ξ)g(Z, R(X, Y,V))]

W_5' (Z,U,R(X,Y,V),ξ) = R'(Z,U,R(X,Y,V),ξ)

= g(U, R(X,Y,V))g(Z,ξ) - g(Z,R(X,Y,V))g(U,ξ)

= R' (X,Y,V, U) η(Z) - R'(X,Y,V,Z)η(U)

putting equations (3.1), (3.2), (3.3) and (3.4) together

η(X) W_5' (Y,Z,U,V) - η(Y) W_5' (X,Z,U,V) - g(U, V)R'(X, Y, Z, ξ) + η(U)R'(X, Y, Z, V) - R' (X, Y, U, V)η(Z) + g(Z,V) R'(X,Y,U,ξ) - R' (X,Y,V, U) η(Z) + R'(X,Y,V,Z)η(U) = 0

Because η (X) and $\eta(Y)$ are non -zero, from the definition of symmetry W_5' (Y,Z,U,V) and W_5' (X,Z,U,V) are zero. Also $\eta(U)$ and $\eta(z)$ are skew- symmetric with respect to the last two terms.
But since g(Z, V) ≠ 0 and g(U,V) ≠ 0 we have R'(X,Y,U,ξ) = 0 and R'(X,Y,Z,ξ) = 0
Hence the theorem.

4. W_5 -Semisymmetric Lorentzian Para- Sasakian Manifold

Definition 4.1: A LP-Sasakian manifold is said to be W_5 -semi-symmetric if

R (X,Y) W_5 (U,V)Z= 0.

Theorem 4.1: A W_5-semi-symmetric LP-Sasakian manifold is W_5 - symmetric LP- Sasakian manifold.

Proof:
Suppose

R(X,Y)W_5(U,V)Z = 0

this implies

g(R(X,Y) W_5(U,V)Z,ξ) =R'(X,Y, W_5(U,V)Z,ξ)

= g(X,ξ)g(Y, W_5(U,V)Z) - g(Y,ξ)g(X, W_5(U,V)Z)

= η(X)W_5'(Y,U,V)Z - η(Y)W_5'(X,U,V)Z

Because η (X) and $\eta(Y)$ are non -zero, it means W_5'(Y,U,V)Z = W_5'(X,U,V)Z=0 but $\nabla_u W_5$(X, Y, Z) = W_5'(X, Y, Z, U) = 0
Hence the theorem.

5. W_5 - Recurrent LP - Sasakian Manifold

Definition 5.1: A LP - Sasakian Manifold is said to be W_5 - recurrent if
∇_u W_5 (X, Y) Z = B(U) W_5 (X, Y) Z

Theorem 5.2: If a LP -Sasakian Manifold is W_5- Recurrent and Ricci -Recurrent, then for the same recurrence parameter its recurrent.
Proof
Given that

W_5' (X, Y, Z, ξ) = R'(X, Y, Z, ξ) + $\frac{1}{n-1}$ [g(X, Z)Ric(Y,ξ) - g(Y,ξ)Ric(X, Z)]

∇_u W_5' (X, Y, Z, ξ) = B(U) W_5' (X, Y, Z, ξ)

∇_u W_5' (X, Y, Z, ξ) = ∇_u R' (X, Y, Z, ξ) + $\frac{1}{n-1}$ [g(X, Z)(∇_uRic)(Y,ξ) - g(Y,ξ)(∇_uRic)(X, Z)]

But (∇_uRic) (Y,ξ) = B(U)Ric(Y,ξ) and (∇_uRic) (X, Z) = B(U)Ric(X, Z)

∇_u W_5' (X, Y, Z, ξ) = ∇_u R' (X, Y, Z, ξ) + $\frac{1}{n-1}$ [g(X, Z) B(U)Ric(Y,ξ) - g(Y,ξ) B(U)Ric(X,Z)] = B(U) W_5' (X, Y, Z, ξ)

Then using Ric(Y,ξ) = (n - 1)g(Y,ξ) and Ric(X,Z) = (n -1) g(X, Z) we get

∇_u W_5' (X, Y, Z, ξ) = ∇_u R' (X, Y, Z, ξ) + $\frac{n-1}{n-1}$ [g(X, Z) B(U)g(Y,ξ) - g(Y,ξ) B(U)g(X,Z)] = B(U) W_5' (X, Y, Z, ξ)

∇_u R' (X, Y, Z, ξ) = B(U) W_5' (X, Y, Z, ξ)

Hence the theorem.

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