PROPERTIES OF UNITARY QUASI-EQUIVALENCE ON SELECTED CLASSES OF OPERATORS IN HILBERT SPACES

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DECLARATION AND RECOMMENDATION

Declaration

This thesis is my original work and has not been presented elsewhere for a degree in any other University.

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Recommendation

This proposal has been submitted for examination with our approval as the university supervisors.

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Dedication

To my Dad, Peter Karanu Thuo & the family, my lovely Wife, Jane Rwaro, and my son Arsen Karanu may you be inspired by this work.

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ABSTRACT

Two operators F and G are considered unitary quasi-equivalent if there exists a unitary operator U satisfying the conditions $F^*F = UG^*GU^*$ and $FF^* = UGG^*U^*$. This concept was introduced in 1996 under the idea of nearly equivalent operators. Since then, various scholars have explored the properties of unitary quasi-equivalence on different operators. For instance, properties of unitary quasi-equivalence on normal, hyponormal, and binomal operators have been investigated. The relationship between unitary quasi-equivalence and other equivalence operators has been established. Specifically, it has been shown that unitary quasi-equivalence implies unitary equivalence. However, the converse is not always true. Partial isometry, co-isometry, isometry, and projection operators have been established to be unitary quasi-equivalence invariants. However, similar properties on the class of w-hyponormal operators, θ-operators, and (p,k)-quasi-hyponormal operators have not been established. This research has therefore, determined the properties of unitary quasi-equivalence on θ-operators using the commutativity concept of an operator. A similar result on w-hyponormal and (p,k)-quasi-hyponormal operators has also been determined in this study using the Aluthge transform and polar decomposition properties. Determining these properties has significant implications in theoretical physics and mathematics. In functional analysis, it contributes to understanding operator algebras and C^*-algebras, impacting their representations, spectra, and K-theory. The outcomes of this research will advance knowledge in interpreting equivalence relations of operators in Hilbert spaces and find practical applications in calculations, wave function differentiation, and the study of vibrations, interfacial waves, and stability analysis. The result of this study shows that unitary quasi-equivalence preserves the properties of θ-operators, w-hyponormal operators, and (p,k)-quasi-hyponormal operators.

Contents

List of Abbreviations

H	Hilbert space
$\mathbb{C}$	Complex numbers
$B(H)$	Banach algebra of bounded linear operators on H
S^*	Adjoint of an operator S
P	Unitary operator
$\|\cdot\|$	Norm of a vector or operator
$\langle\cdot,\cdot\rangle$	Inner product
$\|T\|$	Modulus of an operator T
$\widetilde{S}$	First Aluthge transform of an operator S
$\widetilde{\widetilde{S}}$	Second alugthe Transformation
$\ker(\cdot)$	Kernel of an operator
u.q.e.	Unitary quasi-equivalence

CHAPTER ONE

INTRODUCTION

1.1 Background Information

The historical trajectory of Hilbert spaces intertwines with the development of functional analysis, a branch of mathematics dedicated to studying functions and their properties. Halmos (1982), introduced the Hilbert spaces and named after David Hilbert, who initially explored them in the context of integral equations (Muljadi, 2014). David Hilbert defined a Hilbert space as a distant function that is complete with respect to the norm derived from the inner product. These space serve as a powerful framework for generalizing vector spaces, inner products, and norms to infinite-dimensional settings. Their applications span various fields, including quantum mechanics, Fourier analysis, signal processing, and thermodynamics (Islam, 2020). Since then, a lot of research has been conducted in these spaces. One area of research that has undergone extensive study is equivalence relations with researchers aiming to explore the complex relationships among various types of operators and investigate their fundamental properties. These equivalences are essential in various fields of mathematics especially in functional analysis, operator theory and scientific fields. These equivalence relations in Hilbert spaces include unitary equivalence, metric equivalence, quasi-similarity, and unitary quasi-equivalence (Waihenya, 2014).

The concept of similarity states that operators, F and G, are considered similar if there exists a bounded invertible operator μ such that F can be written as $\mu G\mu^{-1}$. This concept of equivalence maintains several essential properties of operators, such as their eigenvalues and spectra, establishing it as a fundamental aspect of operator theory. On the other hand, a special case of similarity concept is unitary equivalence, obtained by setting operator μ to be unitary, thus guaranteeing the preservation of operator norms. This concept is especially pertinent in the context of self-adjoint operators and has important implications in fields like matrix diagonalization and quantum mechanics. In matrix theory, researchers have investigated the concept of congruence. Congruence preserves essential matrix properties such as characteristic polynomial, determinant, and rank, making it an important tool in matrix analysis.

Fredholm equivalence concerns operators that have a finite difference between the dimensions of their kernel and cokernel. Two Fredholm operators are deemed equivalent if their difference is a compact operator. This idea is crucial in index theory and the study of differential operators. Metric equivalence, as introduced by Nzimbi *et al.* (2012), is an important concept in functional analysis, creating an equivalence relation between bounded linear transformations on a Hilbert space. Two bounded linear operators F and G on a Hilbert space H are metrically equivalent if there exists a constant $C > 0$ such that $\|Fg\| = C\|Gg\|$ for all $g \in H$. This equivalence ensures that the norms of F and G are related by a constant factor, specifically $\|F\| = C\|G\|$, which is crucial for understanding the behavior of operators in terms of their magnitudes. While metric equivalence does not necessarily preserve the spectrum or eigenvalues, it often leads to related spectral properties; for instance, metrically equivalent operators may have similar essential spectra. In some cases, Nzimbi *et al.* (2012), noted that metric equivalence implies unitary equivalence, though the converse is not always true. On the other hand, unitarily equivalent operators are always metrically equivalent, but not necessarily vice versa.

The study of unitary quasi-equivalence was first introduced in 1996 by Ibrahim Othman under the concept of nearly equivalent operator (Nzimbi & Luketero, 2020). Muteti (2014), stated that two operators, say F and G on a Hilbert space H, are nearly equivalent if F^*F and G^*G are similar. Later on, Nzimbi & Luketero (2020) defined unitary quasi-equivalence as follows: Let F and G be bounded linear operators on a Hilbert space H. These operators are considered unitary quasi-equivalent if there exists a unitary operator U such that:

$$F^*F = UG^*GU^*$$

$$FF^* = UGG^*U^*$$

These classes of Hilbert operators has seen alot of research. Nzimbi & Luketero, (2020), established that unitary quasi-equivalence preserves the equivalence relation, making it transitive, symmetric, and reflexive . Unitary quasi-equivalence also preserves the normality and hyponormality of operators. A research by Victor & Nyogesa (2021) established that n-unitary quasi-operators preserve the n-normality, n-posinomality, and n-binomality of operators.

Number of authors has also investigated concept of equivalence relation on different class of operators. Lilian *et al.* (2023), researched on unitary quasi-equivalence and established that it preserves partial isometry, isometry, and co-isometry properties. However, unitary quasi-equivalence does not necessarily preserve skew binormal and skew n-binormal operators but preserves the binormal operator property. Kikete *et al.* (2023), demonstrated that unitary quasi-equivalence preserves the (n,m)-hyponormality of an operator. (Nzimbi & Luketero, 2020) further confirmed the implication that absolute equivalence is also nearly equivalent. Furthermore, they also established that absolute equivalence is weaker than unitary equivalence and unitary quasi-equivalence, i.e.,

Absolute equivalence operators $\subseteq$ Unitary equivalence operators $\subseteq$ Unitary quasi-equivalence operators.

On the same (Nzimbi $ Luketero, 2020) highlighted that for any pair of unitary operators and two isometries within the same Hilbert space, they exhibit metric equivalence, absolute equivalence, and near-equivalence. Musundi *et al.* (2013), noted that unitary quasi-equivalence and metrically equivalence preserves the invertibility of an operator. Nzimbi & Luketero (2020) demonstrated that for orthogonal projection operators F and G in $B(H)$, if the operators are almost unitarily equivalent, then they are unitarily equivalent. Additionally, if two projection operators are unitarily equivalent, it implies they are quasi-equivalent and almost similar.
On the class of θ-operators, Kipkemoi (2016) noted that Normal operator $\subseteq$ θ-operator. Later on, Kipkemoi (2016) also established that θ-operator preserves the quasisimilarity of an operator. Kiprop *et al.* (2021) established that θ-operators also preserve isometry, co-isometry, and unitary equivalence properties of an operator.
The class of hyponormal operators were first examined by Halmos (1950), later on it was explored by Stampfli(1962) using a different nomenclature. Berberian (1962), investigated hyponormal operators, specifically proving that every completely continuous hyponormal operator is a normal operator. This was shown through the use of approximate proper vectors. Aluthge (2000), looked at the properties of p-hyponormal and the relationship that exists with self-adjoint, unitary, and isometry. Otieno (2007) examined the properties of w-hyponormal operators and demonstrated that w-hyponormal operators is a larger class of operators that contains both p and log-hyponormal operators. Nzimbi & Wanyonyi (2020) later contributed to the knowledge of hyponormal operators by establishing that unitary quasi-equivalence operators preserve the hyponomality of an operator as well as the normality, hyponormality, and posinormality of an operators. This exploration of hyponormal operators has greatly contributed to comprehension of Hilbert spaces and their diverse applications thereby making it one of the most significant classes of operators in this field.

Although several research has been done to determine the properties of unitary quasi-equivalence on various operators, there has been less focus on operators such as θ-operators, W-hyponormal operators, and (p,k) hyponormal operators. The primary goal of this study was to establish the properties of unitary quasi-equivalence on specific classes of operators in Hilbert space. Particularly, the study aimed to investigate the properties of unitary quasi-equivalence on θ-operators, W-hyponormal operators, and (p,k) hyponormal operators by leveraging their unique characteristics.

1.2 Statement of the Problem

The properties of unitary quasi-equivalence have been studied for some operators in Hilbert spaces. Unitary quasi-equivalence has been shown to preserve the normality, hyponormality, and binormality of operators. It has also been shown to preserve partial isometry, isometry, and co-isometry of an operator. Although research in this area has revealed extensive applications across numerous fields of mathematics and physics, especially in formulating theories in quantum mechanics, it has not been demonstrated whether certain classes of operators in Hilbert spaces, such as θ-operators, w-hyponormal operators, and (p,k) hyponormal operators, preserve the properties of unitary quasi-equivalence. Therefore, this study aimed to determine the properties of unitary quasi-equivalence in w-hyponormal, θ-operator, and (p,k)-quasi-hyponormal operators.

1.3 Objectives of the Study

1.3.1 General Objective

The main objective of this study was to determine the properties of unitary quasi-equivalence on selected classes of operators in Hilbert spaces.

1.3.2 Specific Objectives

The research will seek to:

(i) Determine the properties of unitary quasi-equivalence on θ-operators.

(ii) Establish properties of unitary quasi-equivalence on w-hyponormal operators.

(iii) Determine the properties of unitary quasi-equivalence on (p,k)-quasi-hyponormal operators.

1.4 Significance of the Study

The study of unitary quasi-equivalence properties on various operators in Hilbert spaces holds significant implications across multiple domains of theoretical physics and mathematics. In the realm of quantum mechanics, this study will offer insights into the characteristics of quantum states and observables, elucidating concepts such as entanglement, coherence, and uncertainty. Furthermore, within functional analysis, the exploration of unitary quasi-equivalence contributes to the comprehension of operator algebras and C*-algebras, impacting their representations, spectra, and K-theory. This research's outcomes will not only advance knowledge in the interpretation of equivalence relations of operators in Hilbert spaces but also find practical applications in wave function differentiation and the study of vibrations, interfacial waves, and stability analysis.

1.5 Definition of Terms

The following basic definitions and concepts are fundamental in this study.

Definition 1.5.1: Vector space (Treves, 2016) . A vector space is defined by two binary operations addition and multiplication on a nonempty set G of objects, called vectors such that it satisfies the following properties, for all $g_1, g_2, g_3 \in G$ and for all scalars ψ, μ:

(i) $g_1 + g_2 \in G$.

(ii) $g_1 + g_2 = g_2 + g_1$.

(iii) $(g_1 + g_2) + g_3 = g_1 + (g_2 + g_3)$.

(iv) There exists an additive identity element $0 \in G$ such that $g_1 + 0 = g_1$.

(v) $\forall\, g_1 \in G$, $\exists -g_1 \in G$ such that $g_1 + (-g_1) = (-g_1) + g_1 = 0$. In this case $-g_1$ is referred to as the additive inverse.

(vi) $\psi g_1 \in G$.

(vii) $\psi(g_1 + g_2) = \psi g_1 + \psi g_2$.

(viii) $(\psi + \mu) g_1 = \psi g_1 + \mu g_1$.

(ix) $(\psi \mu) g_1 = \psi(\mu g_1)$.

(x) $1 \cdot g_1 = g_1$.

Definition 1.5.2: Norm (Okelo, 2020) a **norm** is a function that assigns a strictly positive length or size to each vector in a vector space G, where the zero vector has a length of zero. For a vector space G over the field $\mathbb{R}$ or $\mathbb{C}$, a norm is denoted as $\|\cdot\| : G \to \mathbb{R}$ satisfying these conditions for all $g_1, g_2 \in G$ and scalars α:

(i) $\|\mathbf{g}_1\| \geq 0$, and $\|\mathbf{g}_1\| = 0$ if and only if $\mathbf{g}_1 = 0$.

(ii) $\|\alpha \mathbf{g}_1\| = |\alpha| \|\mathbf{g}_1\|$.

(iii) $\|\mathbf{g}_1 + \mathbf{g}_2\| \leq \|\mathbf{g}_1\| + \|\mathbf{g}_2\|$.

These properties ensure that the norm generalizes the concept of length in Euclidean space. The most familiar example is the Euclidean norm, $\|\mathbf{g}\|_2 = \sqrt{g_1^2 + g_2^2 + \cdots + g_n^2}$, for a vector $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ in $\mathbb{R}^n$. Other examples include the L^1 norm (sum of absolute values), the L^∞ norm (maximum absolute value), and the L^p norms. Norms are fundamental in mathematics, particularly in analysis, linear algebra, and functional analysis, as they quantify vector magnitudes and define distances and convergence.

Definition 1.5.3: A normed space (Kinyanjui *et al.*, 2018). A normed space S is a vector space S together with the norm $\|\cdot\|$ defined on S and is denoted by $(S, \|\cdot\|)$.

Definition 1.5.4: Complete Space (Kinyanjui *et al.*, 2018). A space is said to be complete if there exist a cauchy sequence which converges in it.

Definition1.5.5: An Inner Product Space (Vasudeva & shirali, 2017). An inner product on a vector space U is a function $\langle \cdot, \cdot \rangle : U \times U \to \mathbb{C}$ defined on $U \times U$ with values in $\mathbb{C}$ Satisfying the following:

(i) $\langle g_1, g_2 \rangle \geq 0$ and $\langle g_1, g_1 \rangle = 0$ if and only if $g_1 = 0$ for all $g_1, g_2 \in G$.

(ii) $\langle g_1, g_2 \rangle = \overline{\langle g_2, g_1 \rangle}$ for all $g_1, g_2 \in G$.

(iii) $\langle \alpha g_1 + \beta g_2, g_3 \rangle = \alpha \langle g_1, g_3 \rangle + \beta \langle g_2, g_3 \rangle$ for all $g_1, g_2, g_3 \in G$ and $\alpha, \beta \in \mathbb{C}$.

Definition 1.5.6: Hilbert Space (Kinyanjui *et al.*, 2018). A Hilbert space Is a complete inner product space.

Definition 1.5.7: Operator (Lilian *et al.*, 2023). let F, G be two vector spaces, a mapping S from U to V is said to be an operator if it preserves the underlying structural properties of F and G.

Definition 1.5.8: Linear Operator (Kinyanjui *et al.*, 2018) A mapping $S: U \to V$ is a linear operator if:

(i) $S(u+v) = S(u) + S(v)$, $\forall u, v \in U$.

(ii) $S(\gamma u) = \gamma S(u)$, $\forall u \in U$ and $\forall \gamma \in \mathbb{C}$.

(iii) $S(\gamma u + \alpha v) = \gamma S(u) + \alpha S(v)$, $\forall u, v \in U$ and complex numbers α and γ.

Definition 1.5.9: Bounded operator (Kinyanjui *et al.*, 2018). If $\|Gy\| \leq c\|y\|$ for all $y \in H$ and $c > 0 \in \mathbb{R}$, then the operator $G \in B(H)$ is considered bounded.

Definition 1.5.10: Adjoint Operator (Sobolev, 2017). For a linear operator G on an inner product space F, the adjoint operator G^* is defined such that:

$$\langle G(g_1), g_2 \rangle = \langle g_1, G^*(g_2) \rangle$$

for all vectors g_1 and g_2 in F.

Definition 1.5.11: Properties of Adjoint operator (Huang & Ovsjanikov, 2017). The adjoint operator F^* of F has the following properties:

(i) $I^* = I$, where I is the identity operator.

(ii) $(G+F)^* = G^* + F^*$.

(iii) $(\gamma F)^* = \gamma F^*$, where γ is a scalar.

(iv) $(GF)^* = F^* G^*$.

(v) $(F^*)^* = F$.

(vi) $\|F^*\| = \|F\|$.

(vii) $\|F^* F\| = \|F\|^2$.

Definition 1.5.12: Self Adjoint, Normal, Unitary,Projection and Isometry Operators (Lilian *et al.*, 2023)

Let H be a Hilbert space. An operator $S: H \to H$ is called:

(i) **Self-Adjoint** if $S^* = S$.

(ii) **Normal** if $S^*S = SS^*$.

(iii) **Unitary** if $S^*S = SS^* = I$.

(iv) **Isometry** if $S^*S = I$.

(v) **Co-isometry** if $SS^* = I$.

(vi) **Projection** if $S = S^2$ and $S = S^*$.

(vii) **Partial isometry** if S^*S is a projection.

Definition 1.5.13: Hyponormal Operator (Kinyanjui *et al.*, 2018). An operator $S \in B(H)$ is said to be:

(i) Hyponormal if $S^*S \geq SS^*$.

(ii) Co-hyponormal if $SS^* \geq S^*S$.

Definition 1.5.14: Log-Hyponormal (Prasad & Bachir, 2018). An invertible operator S is called *Log-hyponormal* if $\log(S^*S) \geq \log(SS^*)$.

Definition 1.5.15: Alugthe Transformation (Otieno, 2007).Let $S = U|S|$ represent the polar decomposition of S. We define the **first Aluthge transform** of S as $\widetilde{S} = |S|^{1/2}U|S|^{1/2}$. Additionally, we introduce the **second Aluthge transform** of S as $\widetilde{\widetilde{S}} = |\widetilde{S}|^{1/2}\widetilde{U}|\widetilde{S}|^{1/2}$.

Definition 1.5.16: w-Hyponormal (Rashid, 2017). An operator S is said to be W-hyponormal if it satisfies the condition $|\widetilde{S}| \geq |S| \geq |\widetilde{S^*}|$.

Definition 1.5.17: (Rashid, 2019). If an operator S is w-hyponormal, then:

(i) $\widetilde{S}$ is semi-hyponormal.

(ii) $\widetilde{\widetilde{S}}$ is a hyponormal operator.

Definition 1.5.18: quasi-M-hyponormal (Mecheri & Prasad, 2022). An operator S is said to be quasi-M-hyponormal if there exists a positive real number M such that

$$S^* \left(M^2(S-\gamma)^*(S-\gamma)\right) S \geq S^*(S-\gamma)(S-\gamma)^*S$$

for all $\gamma \in \mathbb{C}$.

Definition 1.5.19:(n,m)**-power hyponormal (Kikete *et al.*, 2023)** Let $S \in B(H)$. Then S is (n,m)-power hyponormal if $S^n(S^m)^* \leq (S^m)^*S^n$ for positive numbers n and m.

Definition 1.5.20: p**-hyponormal (Duggal & Jeon, 2007)** An operator $S \in B(H)$ is defined to be p-hyponormal if $(S^*S)^p - (SS^*)^p \geq 0$ for $0 < p \leq 1$.

Definition 1.5.21: p**-quasihyponormal (Shen & Zuo, 2023).** An operator $S \in B(H)$ is said to be p-quasihyponormal if $S^*((S^*S)^p - (SS^*)^p)S \geq 0$, $0 < p \leq 1$.

Definition 1.5.22: k**-quasihyponormal (Yuan & Wang 2019)** An operator $S \in B(H)$ is said to be k-quasihyponormal if $S^k(S^*S - SS^*)S^k \geq 0$ for $k > 0$.

Definition 1.5.23: (p,k)**-quasihyponormal (Yuan & Wang, 2019).** An operator $S \in B(H)$ is said to be (p,k)-quasihyponormal if $S^{*k}((S^*S)^p - (SS^*)^p)S^k \geq 0$, $0 < p \leq 1$, and a positive integer k (i.e., $k > 0$).

Definition 1.5.24: Equivalence relation (McKenzie, 2019). A binary operation $\sim$ on a set S is said to be an equivalence operation if and only if it satisfies the following conditions:

(i) **Reflexivity:** For every element a $\in S$, $a \sim a$.

(ii) **Symmetry:** If $a \sim b$, then $b \sim a$. $\forall a,b \in S$

(iii) **Transitivity:** If $a \sim b$ and $b \sim c$, then $a \sim c$. $\forall a,b,c \in S$

Definition 1.5.25: Unitary equivalent (Lilian *et al.*, 2023). Two operators $S \in B(H)$ and $T \in B(H)$ are said to be unitarily equivalent if there exists a unitary operator $U \in B(H)$ such that:

$$S = U^*TU,$$

or equivalently:

$$T = U^{-1}SU.$$

Definition 1.5.26: Approximately Equivalent (Nzimbi & Luketero, 2020). Two operators $S \in B(H)$ and $T \in B(H)$ are considered approximately equivalent if there exists a sequence $\{U_n\}$ of unitary operators such that $\{U_n^*TU_n\}$ converges to S in norm

Definition 1.5.27: Similar operator (Mutuku *et al.*, 2023). Two operators $S \in B(H)$ and $K \in B(K)$

are deemed similar, denoted as $S \sim K$, if an invertible operator $U \in B(H,K)$ exists such that:

$$SU = UK$$

or equivalently:

$$S = U^{-1}KU$$

Definition 1.5.28: Quasiaffinity (Mutuku *et al.*, 2023). A quasi affinity operator $S \in B(H)$ is one which is injective and has a dense range.

Definition 1.5.29: Quasisimilar (Mutuku *et al.*, 2023). If there exist quasiaffinities $W \in B(H,K)$ and $Z \in B(K,H)$ such that:

$$WS = TW$$

and

$$WZ = ZT.$$

Then $S \in B(H)$ and $T \in B(K)$ are quasisimilar

Definition 1.5.30: Almost similar operator (Kipkemoi, 2016). Two operators $S,T \in B(H)$ are said to be almost similar if there exists an invertible operator N which satisfies the following properties:

$$S^*S = N^{-1}(T^*T)N$$

$$S^* + S = N^{-1}(T^* + T)N$$

Definition 1.5.31: metrically equivalent operator (Mutuku *et al.*, 2023). For any operators say $S,T \in B(H)$ to be termed as metrically equivalence then:

$$|\langle Sx, Sx\rangle|^{1/2} = |\langle Tx, Tx\rangle|^{1/2}$$

for every $x \in H$, which implies that

$$S^*S = T^*T$$

equivalently, they are metrically equivalent if:

$$\|Sx\| = \|Tx\|$$

Definition 1.5.32: Unitary Quasi-Equivalent Operators (Nzimbi & Luketero, 2020). $(S,T) \in B(H)$ qualifies to be unitary quasi-equivalent if there is a unitary operator U such that:

$$S^*S = UT^*TU^*$$

$$SS^* = UTT^*U^*$$

Definition 1.5.33: n-unitary quasi equivalence operators (Victor & Nyongesa, 2021). Two operators $S,T \in B(H)$ are said to be n-unitarily quasi-equivalent if there exists a unitary operator $U \in B(H)$ such that:

$$S^*S^n = UT^*T^nU^*$$

$$SS^* = UT^nT^*TU^*$$

Definition 1.5.34: Commutativity (Diatta *et al.*, 2022) Operators $S\,T \in B(H)$ commutes if $ST = TS$. The commutativity concept is denoted by $[S,T] = 0$, representing the commutator of S and T.

Definition 1.5.35: Polar decomposition property(Bhunia, 2024) An operator $F \in B(H)$ have a polar decomposition if $F = U|F|$ where U is unitary.

Definition 1.5.36: θ-operator (Kipkemoi, 2016). An operator is defined to be a θ-operator if F^*F and $F+F^*$ commute.

CHAPTER TWO

LITERATURE REVIEW

This chapter consists of three sections in which literature related to this study is reviewed. Useful concepts for subsequent chapters are presented. Specifically, Section 2.1 covers θ-operators, Section 2.2 discusses w-hyponormal operators, and finally, Section 2.3 reviews (p,k)-quasi-hyponormal operators.

2.1 θ-operators

This section present established results on the properties of θ-operators. Additionally, properties of unitary equivalence, metric equivalence, and almost similarity relations within this class of θ-operators were provided.

To understand the properties of this operator the following class inclusion will play a significant role in this section.

$$\text{Normal operators} \subseteq \theta\text{-operators}$$

Kipkemoi (2016), gave a necessary and sufficient condition for an operator to be θ-operator. This is proven in the theorem below.

Theorem 2.1.1: (Kipkemoi, 2016). An operator S which is linear and boundend in Hilbert space H is said to be a θ-operator if F^*F and $F+F^*$ commute. The class of all θ-operators in $B(H)$ is denoted by θ.

Can this property be utilized to demonstrate whether θ-operators preserve the properties of unitary quasi-equivalence? This is an unsolved question that this study aimed to determine.

The investigation of properties of θ-operator on equivalence relation was pioneered by Kiprop, King'ang'i & Khalagai (2021), established that every normal operator is a θ-operator by proving the following theorem.

Theorem 2.1.2: (Kiprop *et al.*, 2021). If $S \in B(H)$ is normal operator then S is also a θ-operator.

In the class of Quasisimilarity operators, Nzimbi *et al.* (2020), investigated properties that are preserved and those that are not preserved through quasisimilarity. Also they established that quasisimilar normal operators are unitarily equivalent, and the implication that almost similarity implies similarity. Also, Nzimbi *et al.* (2020), demonstrated that θ-operators maintain the quasisimilarity of an operator.

The arising question is if the same results can hold for the case of unitary quasi-equivalent. This research therefore aimed to extend the preceding results to the class of unitary quasi-equivalence.

Investigation on the class of almost similarity Kipkemoi (2016) established the following results.

Theorem 2.1.3: (Kipkemoi, 2016). If two operators, say F and G, are almost similar, and G is a θ-operator, then F is also a θ-operator.

Theorem 2.1.4: (Kipkemoi, 2016). For any operators $F, G \in B(H)$ such that $G \in \theta$ and F is almost similar to G, then $F \in \theta$.

Theorem 2.1.5: (Kipkemoi, 2016). If $F \in B(H)$ then $F \in \theta$ if and only if F is almost similar to G whenever G is a normal operator.

On the class of unitary equivalence the following results has been established on the class and sub-classes of θ- operators.

Since normal operators are contained in θ-operators, (Kiprop *et al.*, 2020) established that two unitarily equivalent operators preserve the normality of an operator. This is established in the theorem below:

Theorem 2.1.6: (Kiprop *et al.*, 2020) If two operators $F, G \in B(H)$ are unitarily equivalent, and if F is normal, then G is also normal.

Using the property that unitary operator are either co-isometry or isometry, Kiprop *et al.* (2020), established that θ-operator preserves isometry and co-isometry properties by proving the following theorems.

Theorem 2.1.7: (Kiprop *et al.*, 2020). Let $F, G \in B(H)$ and let F be a θ-operator. if $F = U^*GU$, where U is an isometry. Then, F is also a θ-operator.

Theorem 2.1.8: (Kiprop *et al.*, 2020). Let $F, G \in B(H)$ and let G be a θ-operator. if $F = U^*GU$, where U is a co-isometry. Then, F is also a θ-operator.

Kiprop *et al.* (2021), extended results in Theorems 2.1.7 and 2.1.8 above to the class of unitary equivalence and established that, if two operators are unitary equivalence and one of them is a θ-operator, then the other operator is also a θ-operator. This is evident in theorem 2.1.9 and 2.1.10.

Theorem 2.1.9: (Kiprop *et al.*, 2021). If $F, G \in B(H)$ are two unitary operators such that F is unitary equivalent to G and F is a θ-operator, then G is also a θ-operator.

Theorem 2.1.10: (Kiprop *et al.*, 2021). Let $F, G \in B(H)$ and let F be a θ-operator. Suppose either $F = U^*GU$ or $F = UGU^*$, where U is unitary, . Then, G is also a θ-operator.

The unanswered question is whether θ-operator preserves the unitary-equivalence properties of an operator. This study aimed to determine if F and G are unitary quasi-equivalence and if G is θ-operators then is F also a θ-operator?

Remark 2.1.1: It is now evident that a lot of research has concentrated on determining the unitary equivalence and almost similar equivalence relations within the class and sub-class of θ-operators. However, no research has been conducted on determining the properties of unitary quasi-equivalence on θ-operators. Thus, this research aimed to investigate whether this class of operators preserves the property of unitary quasi-equivalence.

2.2 W-hyponormal operators

This section aims to outline the characteristics of w-hyponormal operators and their relationships within various classes. The following class inclusion hierarchy is pertinent to this investigation:

$$\text{Hyponormal} \subset \text{Log-hyponormal} \subset w\text{-Hyponormal} \subset \text{Paranormal} \subset K\text{-paranormal}$$

$$\text{Hyponormal} \subset p\text{-Hyponormal}\,(0 < p < 1) \subset w\text{-Hyponormal} \subset \text{Paranormal} \subset K\text{-paranormal}$$

Hyponormal operators were introduced by Stampfli (1962) to generalize the concept of a normal operator. An operator is defined to be hyponormal if $F^*F \geq FF^*$. This means that the self-adjoint operator $F^*F - FF^*$ is positive semi-definite. Hyponormal operators attracted the interest of many researchers because they retain some spectral properties of normal operators. These properties include the spectral radius, the isoloid property, and the spectral mapping property. The spectral radius for a hyponormal operator, is equal to the norm. Stampfli (1976) determined the isoloid property of hyponormal operators and realized they are indeed isoloid. Isoloid means that every isolated point in the spectrum belongs to the point spectrum. The spectral mapping theorem, which connects the spectrum of an operator's function to the function of the operator's spectrum, is valid for hyponormal operators.

w-hyponormal operators are a broader class that includes p-hyponormal and log-hyponormal operators. This concept emerged as a further generalization to address certain limitations and explore new operator classes with unique properties. The Aluthge transformation, named after Aluthge, played a significant role in the study of hyponormal operators. From this concept, Otieno (2007) was able to define a w-hyponormal operator F as one satisfying $|\widetilde{F}| \geq |F| \geq |\widetilde{F}^*|$ as shown in theorem 2.2.1 One of the primary areas of interest was the spectral theory of w-hyponormal operators. Researchers aimed to understand how the spectral properties of these operators compare to those of normal and hyponormal operators. Otieno (2007) highlighted that if an operator F is w-hyponormal then $\widetilde{F}$ is semi-hyponormal and $\widetilde{\widetilde{F}}$ is a hyponormal operator. Duggal & Jeon (2004) noted that a p-hyponormal spectral operator is normal. Jeon *et al.,*(2004) extended the study result of Sheth (1966), who demonstrated that the spectral operator is normal in the case of M-hyponormal operators. Also, Stampfli (1976) had established that a dominant operator may be quasinilpotent and hence spectral, without necessarily being normal. Otieno (2007) investigated the spectral properties of W-hyponormal operators and noted that a W-hyponormal operator and a spectral operator are related in a specific way through an operator with a dense range. The W-hyponormal operator not only becomes normal but also aligns the spectral operator as shown in Theorem 2.2.4. This implies that the spectral structure is preserved and simplified, showing a strong connection between the spectral characteristics of the W-hyponormal operator and the associated spectral operator.

Otieno (2007) gave a conditions under which an operator should satisfy to be be a w-hyponormal. This was showed by proving theorem 2.2.1.

Theorem 2.2.1: (Otieno 2007). an operator $F \in B(H)$ is said to be a w-hyponormal if it satisfy the condition $|\widetilde{F}| \geq |F| \geq |\widetilde{F}^*|$, where $|\widetilde{F}|$ denotes the Alugthe transformation.

The undetermined question is whether this condition can be used to establish some properties of unitary quasi equivalence on W-hyponormal operator?.

However, based on the w-hyponormal properties above, numerous attempts have been made to identify the properties of various equivalence relations within this class of operators. The following results have been established.

Duggal & Jeon (2007), showed that the normal components of quasi-similar p-hyponormal operators are unitarily equivalent. They also established that a p-hyponormal operator, if compactly quasi-similar to an isometry, is unitary. Duggal & Jeon (2007), also realized that a p-hyponormal operator which is a spectral operator is determined to be normal. This was presented in theorem below

Theorem 2.2.2: (Duggal & Jeon, 2007).

(i) The operator $F \in p$-hyponormal is normal if and only if the corresponding operator G is normal, and in this case, $F = G$.

(ii) If $F \in p$-hyponormal and $\sigma(F) \subseteq ND$, then F is unitary.

Theorem 2.2.3: (Jeon *et al.*, 2004). Consider G as a p-quasihyponormal operator. If the restriction $G|M$ of G to an invariant subspace M is a one-to-one normal operator, then G is reduced by M. Aluthge's (2007) research on p-hyponormal operators indicated that for any p-hyponormal operator with $p > 1$, the operator is also hyponormal. However, the emphasis has been placed on p-hyponormal operators for $0 < p \leq 1$. This study will also leverage the open interval in determining the unitary quasi-equivalence attributes of the corresponding operator. Under this interval several establishments has been made. The following are some of results in form of a theorem.

Theorem 2.2.4: (Fujii, 2003) If F is an arbitrary p-hyponormal operator in $B(H)$ for some $p \in \left[\frac{1}{2}, 1\right]$ with a polar decomposition $F = U|F|$, then the Aluthge transform $\widetilde{F}$ is hyponormal.

Jung (2003), also noted that Aluthge transform does not rely on partial isometry of an operator. This result was presented in theorem 2.2.4 below:

Theorem 2.2.5: (Fujii, 2003). Let $G = \mu|G|$ be the polar decomposition of G. If there exists another decomposition $G = \psi|G|$, then $\widetilde{G} = |G|^{\frac{1}{2}}\mu|G|^{\frac{1}{2}} = |G|^{\frac{1}{2}}\psi|G|^{\frac{1}{2}}$.

Can this property be used to show if unitary quasi-equivalence preserves the w-hyponomality of an operator?. This study aimed to determine this gap.

More interestingly, this results were extendend and summarized in thorem 2.2.6 and 2.2.7as follows;

Theorem 2.2.6: (Duggal & Jeon, 2007). Let $G \in B(H)$ be p-hyponormal. Then

i. If $p > 1$, then $\widetilde{G}$ is hyponormal,

ii. If $p < 1$, then $\widetilde{G}$ is $\left(\frac{p+1}{2}\right)$-hyponormal,

iii. $\widetilde{\widetilde{G}}$ is hyponormal.

Theorem 2.2.7: (Duggal & Jeon, 2007). For a polar decomposition $G = \mu|G|$ where $1 \geq p > 0$, the operator $\widetilde{G} = |G|^q\mu|G|^q$ is hyponormal for any q in the interval $p \geq q > 0$.

Duggal & Jeon (2007) continued to make several significant contributions to the understanding of p-hyponormal operators. They established that the set of p-hyponormal operators is closed under addition and scalar multiplication, demonstrating that if G_1 and G_2 are p-hyponormal operators and α is a scalar, then $G_1 + G_2$ and αG_1 are also p-hyponormal. Duggal & Jeon (2007) also demonstrated that the spectrum of p-hyponormal operators lies in the closure of the numerical range. This was presented in theorems below:

Theorem 2.2.8: (Duggal & Jeon, 2007). If G_1 and G_2 are p-hyponormal operators on a Hilbert space, then $G_1 + G_2$ is also a p-hyponormal operator.

Theorem 2.2.9: (Duggal & Jeon, 2007). If G is a p-hyponormal and α is a scalar, then αG is also a p-hyponormal operator.

In the same research, Duggal & Jeon (2007) investigated the behavior of p-hyponormal operators under functional calculus. They established that if G is a p-hyponormal operator and f is a polynomial, then $f(G)$ is also p-hyponormal. They also showed that if a p-hyponormal operator G commutes with a normal operator M, their product GM remains p-hyponormal. These results were supported by the commutation relations and properties of normal operators. The results represented in form of a theorem as follows:

Theorem 2.2.10: (Oloomi & Radjabalipour, 2012). If G is p-hyponormal and M is a normal operator on a Hilbert space such that $GM = MG$, then GM is also a p-hyponormal operator.

Will the same result hold in the case of unitary quasi-equivalence? This study aimed to show if unitary quasi-equivalence preserves the p-hyponomality of an operator.

Remark:

From the preceding results on p-hyponormal operators indicate that the normal parts of quasi-similar p-hyponormal operators are unitarily equivalent. Additionally, every p-hyponormal operator that is compactly quasi-similar to an isometry is, in fact, unitary. However, a similar result has not been demonstrated for the class of unitary quasi-equivalence. Therefore, this study aimed to address this gap.

Similarity, on the class of log-hyponormal operator Jeon *et al*. (2004), established the invertibility criteria, the quasisimilar and the normality of log-hyponormal in relation to the unitarily equivalent. This was shown by proving the following theorem.

Theorem 2.2.11: (Jeon *et al*., 2004). Let B be an invariant subspace of a log-hyponormal operator $G \in L(H)$, and let $G|_B$ be the restriction of G to B. If $G|_B$ is invertible, then $G|_B$ is log-hyponormal.

will this property hold in case of unitary quasi-equivalence? This study aimed to establish this gap.

Theorem 2.2.12: (Jeon *et al*., 2004). If $G \in B(H)$ is a log-hyponormal operator, then G can be expressed as $G = G_1 \oplus G_2$ on the space $H = H_1 \oplus H_2$, where G_1 is normal and G_2 is pure and log-hyponormal. Additionally, G_2 does not have any invariant subspace B such that $G_2|_B$ is normal.

Theorem 2.2.13: (Jeon *et al*., 2004). Let $G_1 \in B(H_1)$ and $G_2 \in B(H_2)$ be a log-hyponormal operator and a normal operator respectively. If there is an operator $F \in B(H_2, H_1)$ with a dense range such that $G_1 F = F G_2$, then G_1 is normal.

Theorem 2.2.14: (Jeon *et al.*, 2004). Let $G_i \in B(H_i)$ for $i = 1, 2$ be log-hyponormal operators, decomposed as $G_i = P_i \oplus K_i$ on $H_i = H_{i1} \oplus H_{i2}$, where P_i and K_i are the normal and pure components of G_i, respectively. If G_1 and G_2 are quasisimilar, then P_1 and P_2 are unitarily equivalent.

Theorem 2.2.15: (Jeon *et al.*, 2004). Let $G_1 \in B(H_1)$ be log-hyponormal and $G_2 \in B(H_2)$ be an isometry. If G_1 and G_2 are quasisimilar operators, it implies they are also unitarily equivalent unitary operators.

Otieno (2007) later expanded this understanding to a broader class of operators known as w-hyponormal operators. Significantly, Otieno (2007) discovered that these operators conform to the properties of unitarily equivalent normal operators. Under hyponormal operator category, Otieno, (2007) demonstrated that hyponormal operators that are quasi-similar are unitarily equivalent. Jeon and Duggal, extended Conway's result to the classes of p-hyponormal and log-hyponormal operators, respectively, and established that the operators are both quasi-similar and unitarily equivalent.

Theorem 2.2.16: (Otieno, 2007). Let $G = \mu|G|$ be a w-hyponormal operator. If $G = |G|^2\mu|G|^2$ is normal, then G is also a normal operator.

Aluthge established some properties which contributed greatly in definition of w-hyponormal and some distinctive attributes of it. Also Aluthge observed that the transformation $G = \mu|G|$ is independent of the partial isometry used in the decomposition of an operator. This properties were established in theorem below.

Theorem 2.2.17: (Mutuku, 2020). If $S \in B(H)$. Then:

(i.) $\alpha\ \widetilde{(S)} = \alpha(\widetilde{S})$ for all $\alpha \in \mathbb{C}$.

(ii.) $\widetilde{(\mu S \mu^*)} = \mu(\widetilde{S})\mu^*$.

(iii.) If $S = S_1 + S_2$, then $\widetilde{S} = \widetilde{S_1} + \widetilde{S_2}$.

(iv.) $\|\widetilde{S}\|_2 \leq \|S\|_2$.

(v.) $\sigma(\widetilde{S}) = \sigma(S)$.

Theorem 2.1.18: (Mutuku, 2020). For an operator $G \in B(H)$ on a Hilbert space H, with $G = U|G|$ and $\widetilde{G}$ as the Aluthge transform of G, the following statements are true:

(i) $\sigma(G) = \sigma(\widetilde{G})$

(ii) $\sigma_p(G) = \sigma_p(\widetilde{G})$

(iii) $\sigma_{ap}(G) = \sigma_{ap}(\widetilde{G})$

(iv) $\sigma_e(G) = \sigma_e(\widetilde{G})$

(v) $\sigma_{le}(G) = \sigma_{le}(\widetilde{G})$

(vi) $\sigma_{re}(G) = \sigma_{re}(\widetilde{G})$

(vii) $\|\widetilde{G}\| \leq \|\widetilde{G^2}\|^{1/2} \leq \|G\|$

Remark: The assertions (i) through (vi) correspond to various spectral properties of the operator G and its Aluthge transform $\widetilde{G}$. Specifically, (i) indicates the equivalence of their spectra, (ii) denotes the equality in their point spectra, (iii) shows agreement in their approximate point spectra, (iv) confirms identical essential spectra, (v) and (vi) respectively match their left and right essential spectra. Finally, (vii) provides a norm inequality relation involving G and $\widetilde{G}$, illustrating a fundamental property of their transformations.

Aluthge & Wang (2003) investigated the properties under which an operator qualifies as a w-hyponormal operator. They found that if the operator's numerical radius satisfies the condition $|\widetilde{G}| \geq |G| \geq |\widetilde{G}^*|$, then it is w-hyponormal. More results on this are substantiated in Theorem 2.2.20 below.

Theorem 2.2.19: (Mutuku, 2020). If an operator say G is w-hyponormal, then its second Aluthge operator $\widetilde{G}$ is semi-hyponormal, and if G is semi-hyponormal, then $\widetilde{G}$ is hyponormal. On the kernel condition of w-hyponormal, Mutuku (2020) determined that w-hyponormal operators does not satisfy the well-known kernel condition $\ker(G) = \ker(G^*)$, which holds whenever G is normal. However, the kernel condition can hold when additional requirements are imposed on w-hyponormal operators. Mutuku (2020), presented this condition by proving Theorem 2.2.21 below:

Theorem 2.2. 20: (Mutuku, 2020). Let G be w-hyponormal with $\ker(G) \subseteq \ker(G^*)$. If $\widetilde{G}$ is normal, then $G = \widetilde{G}$. This means that the kernel condition for w-hyponormal operators holds only when the operator $\widetilde{G}$ is normal.

On the class of unitary equivalence operator, Otieno (2007) established the following results

Theorem 2.2.21: (Otieno 2007). If F and G^* are w-hyponormal operators with $\ker F \subset \ker F^*$ and $\ker G^* \subset \ker G$, and if there exists a quasiaffinity operator U such that $U \in \ker \delta F, G$, then F and G are unitarily equivalent normal operators.

An unanswered question is whether, if two operators F and G are unitary quasi-equivalence operators, and F is w-hyponormal, is G necessarily w-hyponormal?. However, The ressearch aimed to determineo if $F \overset{\text{u.q.e.}}{\sim} G$ preserves the w-hyponormality of operator.

On the relationship between unitary equivalence and Quasi similar on the class of W-hyponormal operators, (otieno, 2007), established that quasi-similar normal operators are also unitary equivalent and proved in the theorem 2.2.23.

Theorem 2.2.22: (Otieno, 2007). If a w-hyponormal operator F with $\ker F \subseteq \ker G^*$ is quasisimilar to a normal operator G, then F and G are unitarily equivalent.

Does a unitary equivalence operator preserve the unitary-quasi equivalence property of an operator? This study aims to address and establish this gap.

Theorem 2.2.23: (Otieno, 2007): Let F be a w-hyponormal operator with $\ker F \subset \ker F^*$, and G be a spectral operator. If there is an operator $f \in B(H)$ with a dense range such that $f \in \ker \delta_{F,G}$, then F is normal, G is scalar, and G is similar to F.

On the class of unitary quasi-equivalence, Muteti (2014) established that unitary equivalence implies unitary quasi-equivalence. However, Muteti (2014) noted that unitary quasi-equivalence does not imply unitary equivalence. It was noted to hold only only when the operators are similar normal operators (Muteti, 2014). These results were determined in the Theorems below:

Theorem 2.2.24: (Muteti, 2014). If $F, G \in B(H)$ are unitarily equivalent operators, then they are unitarily quasi-equivalent.

In addition to that, Muteti (2014) established that unitary quasi-equivalence preserves the normality of an operator, and the result is presented in the theorem below:

Theorem 2.2.25: (Muteti, 2014). If $F, G \in B(H)$ are unitarily quasi-equivalent operators, then F is normal if and only if G is normal.

Nzimbi & Wanyonyi(2020), extedend the results in theorem 2.2.6 above to other class of operators in the hilbert space. They established that unitary-quasi equivalence preserves the hyponormality of an operator as presented in the theorem below.

Theorem 2.2.26: (Nzimbi & Wanyonyi, 2020). Let $F, G \in B(H)$ be unitarily quasi-equivalent operators,then F is hyponormal if and only if G is hyponormal.

Rashid (2020), established that the following statement in theorem 2.2.28 and 2.2.29 are equivalent;

Theorem 2.2.27: (Rashid, 2019). Suppose S belongs to the set of bounded operators on a Hilbert space H. Then the following conditions are equivalent:

(i) S is p-w-hyponormal;

(ii) $|S|^p \geq \left(|S|^{\frac{1}{2}}|S^*||S|^{\frac{1}{2}}\right)^{\frac{p}{2}}$ and $\left(|S^*|^{\frac{1}{2}}|S||S^*|^{\frac{1}{2}}\right)^{\frac{p}{2}} \geq |S^*|^p$;

(iii) $|\widetilde{S*}^*|^p \geq |S^*|^p \geq |\widetilde{S*}|^p$.

Theorem 2.2.28: (Rashid, 2019). Let $G \in B(H)$. The following conditions are equivalent:

(i) G is (s,p)-w-hyponormal;

(ii) $|G|^{2sp} \geq (|G|^s|G^*|^{2s}|G|^s)^{\frac{p}{2}}$ and $(|G^*|^s|G|^{2s}|G^*|^s)^{\frac{p}{2}} \geq |G^*|^{2sp}$;

(iii) $|\widetilde{G*}(s,s)^*| \geq |G^*|^{2sp} \geq |\widetilde{G*}(s,s)|$.

However it has not been shown if unitary-quasi equivalence preserves the w-hyponormality of an operator. this still remain to be unchallenged question.

Luketero (2020), later extendend this results on isometry and co-isometry properties of hyponormal operator. Luketero (2020), noted that hyponormal operator are isometric and co-isometric invariant by proving the following theorem 2.2.5:

Theorem 2.2.29: (Luketero & Khalagai, 2020). Let F be a hyponormal operator and G be another operator such that:

(i) $F = UGU^*$ where U is an isometry, or

(ii) $F = U^*GU$ where U is a co-isometry.

Then G is also hyponormal.

Remark 2.2.1: Studies have explored the properties of W-hyponormal operators and examined the characteristics of unitary equivalence within subclasses of W-hyponormal operators. However, the investigation into the properties of unitary quasi-equivalence on W-hyponormal, log-hyponormal and p-hyponormal operators has not been determined. Therefore, there is a need for research in this specific area. This study aims to determine whether unitary quasi-equivalence maintains the W-hyponormality, log-hyponormality and p-hyponormality of operators.

2.3 (p,k)-quasi-hyponormal operators

In this section, the history trajectory of (p, k)-quasi-hyponormal along with its distinctive properties is presented. Hyponormal operators were introduced in 1962 to generalize normal operators (Stampfli, 1962). In this case, every normal operator was regarded as hyponormal due to the relaxed inequality condition in their definition (Stampfli, 1962). Later, Campbell (1978) proposed k-quasihyponormal operators to be a broader class of hyponormal operators and demonstrated that any hyponormal operator is k-quasihyponormal when $k = 1$. Subsequently, Aluthge (1990), expanded on the concept of hyponormal operators by introducing p-hyponormal operators, noting that choosing $p = 1$ in the definition results in a hyponormal operator. Similarly, setting $p = 1$ in the definition of p-quasihyponormal, Aluthge (1990), established that the result includes the set of all quasihyponormal operators.

Furuta (1998), introduced class A operators, a broader category that includes p-hyponormal operators. Tanahashi (1999) introduced an operator which contained the set of all invertible hyponormal operator. This operator was named log-hyponormal (Tanahashi, 1999). Tanahashi (1999) also worked on invertible p-hyponormal operators, discovering that they are indeed log-hyponormal. Furthermore, Tanahashi (1999) determined that this class of operators belongs to the class A operators. Fujii *et al.* (2000), subsequently broadened the scope of class A operators by defining class A(s, t) and further extending it to absolute-(s, t) paranormal operators. They also introduced class AI(s, t), which was a class that contained all invertible operators. Aluthge *et al.* (2000), broadened the scope of log- and p-hyponormal operators to a larger class called the w-hyponormal operators. This was achieved through the use of Aluthge transformations. Still in this research, Aluthge *et al.* (2000), prove that every semi-hyponormal operator is w-hyponormal and further contained all p-hyponormal operators for any $p \geq 1$.

The class of $wA(s,t)$ operators was introduced by Ito (2001) as a broader concept that generalizes w-hyponormality. By specifically choosing $s = 1$ and $t = \frac{1}{2}$ in the definition of $wA(s,t)$ operators, one can derive the class of w-hyponormal operators. This guarantees that the w-hyponormal operator is a special case of wA(s, t) operators and thus a containment in that class. From the class of wA(s, t) another special class of operator is derived. This class is referred to as wA(1, 1) or simply called wA. This implies that an operator G belongs to the class wA if and only if $|G^2| \geq |G|^2$ and $|G^*|^2 \geq |G^{*2}|$. However, given the definition of class A operators, it is evident from the property $|G^2| \geq |G|^2$ that class A encompasses class wA. Further analysis by Ito (2001) depicts that the class wA(s1, t1) is contained within wA(s2, t2) where $s2 \geq s1$ and $t2 \geq t1$. This implies that $wA\left(\frac{1}{2}, \frac{1}{2}\right) \subseteq wA(1,1)$, which serves as a broader concept of w-hyponormality because $wA\left(\frac{1}{2}, \frac{1}{2}\right)$ is associated with w-hyponormal operators, while $wA(1,1)$ is related to class A operators.

Moving forward, another class denoted as A(k) was introduced as a generalization of class A, with all class A operators being A(k) when $k = 1$, thus satisfying $|G^2| \geq |G|^2$ (Yanagida, 2003). The class A(s, t) operators show that class A(k, 1) equals class A(k), which is a subclass of A(s, t). The class of (p, k)-quasihyponormal was generalized from the concept of p-quasihyponormal and k-quasihyponormal by Kim (2003), with $p = 1$ and $k = 1$ yielding k-quasihyponormal and p-quasihyponormal operators, respectively. Kim (2003) also demonstrated that q-quasihyponormal operators are (p, k)-quasihyponormal since $0 < q < p$ and that (p, k)-quasihyponormal operators include p-hyponormal operators. Jibril (2008) extended the concept of (p, k)-quasihyponormal by introducing 2-Power normal operators and later generalized them into n-Power normal operators. Ahmed (2011) extended the results of Jibril by further generalizing 2-Power normal operators into n-Power quasinormal operators, while Panayan & Sivaman (2012), introduced n-Power Class(Q) operators as an extension of all normal operators.

Tanahashi *et al.* (2004) introduced another class, k-λ class paranormal operators, which contains all invertible hyponormal operators. Huru showed that invertible p-hyponormal operators are k-hyponormal, and class k-hyponormal operators are class A operators.

from this history the following class inclusion is evidence;

(i.) spectroid $\subset$ hen-spectroid $\subset$ numeroid $\subset$ transaloid $\subset$ normaloid $\subset$ spectraloid.

(ii.) ∞-hyponormal $\subset$ normal $\subset$ n-powernormal $\subset$ n-power-quasinormal.

(iii.) self-adjoint $\subset$ normal $\subset$ hyponormal $\subset$ p-hyponormal $\subset$ p-quasihyponormal $\subset$ (p,k)-quasihyponormal.

(iv.) hyponormal $\subset$ transaloid $\subset$ convexoid.

(v.) hyponormal $\subset$ quasihyponormal $\subset$ k-quasihyponormal $\subset$ (p,k)-quasihyponormal.

(vi.) spectroid $\subset$ hen-spectroid $\subset$ numeroid $\subset$ transaloid $\subset$ convexoid.

(vii.) hyponormal $\subset$ quasihyponormal $\subset$ k-quasihyponormal $\subset$ (p,k)-quasihyponormal.

(viii.) ∞-hyponormal $\subset$ k-hyponormal $\subset$ p-hyponormal $\subset$ w-hyponormal $\subset$ class $AI(s,t)$ $\subset$ class $wA(s,t)$ $\subset$ class $A(s,t)$.

(ix.) normal $\subset$ hyponormal $\subset$ p-hyponormal $\subset$ normaloid $\subset$ HN.

(x.) p-hyponormal $\subset$ semi-hyponormal $\subset$ w-hyponormal $\subset$ wA $\subset$ class A $\subset$ class $A(k)$ $\subset$ class $A(s,t)$.

(xi.) CTHN $\subset$ THN $\subset$ HN.

(xii.) p-hyponormal $\subset$ semi-hyponormal $\subset$ w-hyponormal $\subset$ wA $\subset$ class $A(s,t)$ $\subset$ class $A(s,t)$.

(xiii.) ∞-hyponormal $\subset$ normal $\subset$ quasinormal $\subset$ n-power-quasinormal.

(xiv.) subnormal $\subset$ hyponormal $\subset$ quasihyponormal $\subset$ class(A) $\subset$ paranormal.

(xv.) $\log$-hyponormal $\subset$ w-hyponormal $\subset$ class $AI(s,t)$ $\subset$ class $wA(s,t)$ $\subset$ class $A(s,t)$.

In particular, for this case the foollowing class inclusion will play a crucial role in examining (p,k)-quasi-hyponormal operators:

hyponormal operators $\subseteq$ p-hyponormal operators $\subseteq$ p-quasi-hyponormal operators $\subseteq$ (p,k)-quasi-hyponormal ope

hyponormal operators $\subseteq$ k-quasi-hyponormal operators $\subseteq$ (p,k)-quasi-hyponormal operators

Kikete *et al.* (2023), defined (p,k)-quasi-hyponormal operators as follows.

Theorem 2.3.1 (Kikete *et al.*, 2023). Operator $F \in B(H)$ is (p,k)-quasihyponormal if $F^{*k}((F^*F)^p - (FF^*)^p)F^k \geq 0$, $0 < p \leq 1$, and a positive integer k.

More specifically, this class of hyponormal operator becomes p-hyponormal operators if $k = 0$, k-quasi-hyponormal operators if $p = 1$, and p-quasi-hyponormal operators if $k = 1$) (Kiketo *et al.*, 2005).

However, the undetermined question is whether this properties can be used to establish some properties of unitary quasi-equivalence on (p,k)-quasi-hyponormal operator?. This study aimed to establish this gap.

Kikete *et al.* (2023) introduced a new category of operators called (n,m)-hyponormal operators and introduced the concept of (n,m)-quasiequivalence, while demonstrating it as an equivalence relation. Various characteristics of (n,m)-hyponormal operators were established. More specifically it was demonstrated that if an operator G is (n,m)-hyponormal, and operator F is (n,m)-quasiequivalent to G, then G also qualifies as an (n,m)-hyponormal operator. This results were shown in the following theorems.

Theorem 2.3.2: (Kikete *et al.*, 2023). Suppose that G is an (n,m)-hyponormal operator, and F is (n,m)-quasiequivalent operator to G. Then F is as an (n,m)-hyponormal operator.

Kikete *et al.*, (2023), established that this class of operator preserves the unitary equivalence as well as unitary quasi-equivalence properties of an operator. Additionally, unitary equivalence (n,m)-hyponormal operators are unitary quasi-equivalence (n,m)-hyponormal operators (Kikete *et al.*, 2023). It was also established that if an operator, say $F = UGU^*$ where U is unitary, then F is also (n,m)-hyponormal. Similarly, when U is co-isometry, then F is also (n,m)-hyponormal operator. The results were presented in the theorem below.

Theorem 2.3.3: (kikete *et al.*, 2023). Suppose $F \in B(H)$ is an (n,m)-hyponormal operator. Let U and V be an isometry and a co-isometry, respectively. Then for an operator $G \in B(H)$:

(i) If $G = UFU^*$, then G is also (n,m)-hyponormal.

(ii) If $G = V^*FV$, then G is also (n,m)-hyponormal.

The question that arises is whether the concept of co-isometry and unitary properties under the unitary quasi-equivalence concept can be utilized to derive certain results in (p,k)-quasihyponormal operators? This is still an open area that has not been done so far. In relation to power hyponormal operators, Kikete *et al.*,(2023) also established the following results.

Theorem 2.3.4: (Kikete *et al.*, 2023). Two bounded linear operators F and G in a Hilbert space H are considered to be (n,m)-unitarily quasi-equivalent if there exists a unitary operator U such that $F^n \cdot F^m = U \cdot G^n \cdot G^m \cdot U^*$ and $F^m \cdot F^n = U \cdot G^m \cdot G^n \cdot U^*$.

Can this properties be used to show whether (p,k)-quasi-hyponormal operators preserves the unitary equivalence operator? This study aimed to answer this question.

Remark 2.2.3:

From this section it is evidence that the properties of unitary quasi-equivalence on (p,k)-quasi-hyponormal operator has not been established. Therefore, this research aimed to establish this gap.

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Introduction

This section presents the approach that has been undertaken to successfully meet the objectives of the study. Important theorems that have necessitate the proofs of objectives have also been presented. In addition, this section presents the ethical considerations observed throughout the study.

3.2 Research Methods

To successfully complete this study, the background information on Functional Analysis and Operator Theory have played an important role. Throughout the course of the study, various techniques to ascertain the characteristics of unitary quasi-equivalence in relation to the θ-operator, the properties of w-hyponormal operators, and the attributes of (p,k)-quasi-hyponormal operators have been presented.

3.2.1 Determining Properties of unitary Quasi-Equivalence on θ-operator

Inorder to meet this objective the properties of θ-operator will be utilized. Additionally, properties of almost similar and properties of unitary equivalence on θ-operator will also be used for comparison. In general, this study aims to extend the following general and technical

3.2.1.1 General properties of θ-operator

To clearly meet the objective one, understanding the following basic properties of θ-operator will be paramount. This general properties are as follows;
An operator $G \in B(H)$ is defined to be a θ-operator if G^*G commutes with $G^* + G$.
However, for an operator $D \in \theta$,

$$4D^*D - (D^* + D)^2 \geq 0.$$

If we define $G = D^* + D + i\sqrt{\frac{4D^*D-(D^*+D)^2}{2}}$. Then G is normal and $\sigma(D)$ is contained in the closed upper half-plane. From this it can be inferred that:

$$G^*G = D^*D \quad \text{and} \quad G^* + G = D^* + D.$$

Therefore, for an operator $D \in B(H)$, if D is self-adjoint, then D^*D commutes with $D^* + D$. This can be demonstrated as follows:

$$(D^*D)(D^* + D) = (D^* + D)(D^*D).$$

Hence,

$$(D^*D)(D^* + D) = (D^*DD^* + D^*DD).$$

Let $D^*D = F$. Then we have

$$D^*DD^* + D^*DD = FD^* + FD = (D^* + D)F = (D^* + D)(D^*D).$$

However this property can be well presented using the following example;

Example Consider an operator $F = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix}$ on the two-dimensional Hilbert space H_2. demonstrate that F is a θ-operator.

Solution:

Since F is self-adjoint, then $F^* = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix}$. For the θ-operator property, we need to establish if $(F^*F)(F^* + F) = (F^* + F)(F^*F)$. So,

$$F^*F = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 20 & 32 \\ 32 & 52 \end{pmatrix}$$

And

$$F^* + F = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 8 & 12 \end{pmatrix}.$$

Thus,

$$(F^*F)(F^* + F) = \begin{pmatrix} 20 & 32 \\ 32 & 52 \end{pmatrix} \begin{pmatrix} 4 & 8 \\ 8 & 12 \end{pmatrix} = \begin{pmatrix} 336 & 512 \\ 512 & 832 \end{pmatrix}$$

and

$$(F^* + F)(F^*F) = \begin{pmatrix} 4 & 8 \\ 8 & 12 \end{pmatrix} \begin{pmatrix} 20 & 32 \\ 32 & 52 \end{pmatrix} = \begin{pmatrix} 336 & 512 \\ 512 & 832 \end{pmatrix}.$$

Hence, $(F^*F)(F^*+F) = (F^*+F)(F^*F)$.

Theorem 3.2.1.1.1 (Kiprop, 2016): Every normal operator $G \in B(H)$ is a θ-operator.

Proof:

Given that G is normal, we have $G = GG^*G$. Based on the properties of a θ-operator, we get

$$G^*G = G^*G(G^*+G)$$

$$= G^*GG^* + G^*GG \quad (1)$$

Additionally,

$$(G^*+G)G^*G = G^*G^*G + GG^*G \quad (2)$$

From the right-hand side of equation (1), we get

$$G^*GG^* + G^*GG = G^*G^*G + G^*GG$$

$$= (G^*)^2G + G^*G^2 \quad \text{(Due to the commutativity of } G \text{ and } G^*\text{)}$$

$$= (G^*)^2G + GG^*G,$$

which corresponds to the right-hand side of equation (2).

Therefore, every normal operator is a θ-operator.

Theorem 3.2.1.1.2: (Muteti, 2014). If $F, G \in B(H)$ are such that $G \in \theta$ and F is almost similar to G, then $F \in \theta$.

Proof:

Since F is almost similar to G, it Implies that, an invertible operator D exists such that

$$F^*F = D^{-1}G^*GD^* \tag{1}$$

and

$$F^* + F = D^{-1}(G^*+G)D \tag{2}$$

equation (1) can be extendend and rewritten as

$$F = FD^{-1}G^*GD^* = FD^{-1}DG^*G = FG^*G$$

and thus, $F^* = (FG^*G)^* = G^*GF^*$.

Applying the properties of θ-operator, the following results are obtained;

$$F^*F = G^*GF^*F = G^*G(GF^*)G = G^*GG^*G = (G^*G)^2 = G^*G \quad \text{(Projection property)}$$

Also,

$$D^{-1}(G^* + G)D = F^* + F = G^*GG^* + GG^*G$$

But S is almost similar to T hence;

$$\begin{aligned} F^* + F &= G^*GG^* + G^*GG^* \\ &= G^*GU^*UG^*G + U^*GG^*U \\ &= G^*G(G^*G)^* + G(G^*G)^* \\ &= G^*G + GG^* \end{aligned}$$

This is an indication that G is also a θ-operator.

Theorem 3.2.1.1.3: (Campbell, 1980). Let $F, G \in B(H)$. If F is normal and unitarily equivalent to G, then G is normal.

Proof:

By hypothesis, F is normal which implies that $G = U^*FU$, where U is unitary. From this, it follows that:

$$G^*G = (U^*F^*U)(U^*FU) = U^*F^*FU = U^*FF^*U = GU^*F^*U = GU^*UG^* = GG^*.$$

This proves the claim.

Theorem 3.2.1.1.4: (Kiprop *et al.*, 2020). If two operators $F, G \in B(H)$ are unitarily equivalent, and if F is normal, then G is also normal.

Proof:

Suppose that F is unitary quasi-equivalence and G is a normal operator. Then from the properties of unitary quasi-equivalence, we can write F and G as follows:

$$F^*F = PG^*GP^* \quad \text{and} \quad FF^* = PTT^*P^*$$

This further implies that:

$$F^*F = PG^*GP^* = PGG^*P^* = FF^*$$

Conversely:

$$FF^* = PGG^*P^* = PG^*GP^* = F^*F$$

This implies that $FF^* = F^*F$, hence S is normal.

Theorem 3.2.1.1.5: (Kiprop *et al.*, 2021). Let $F, G \in B(H)$ and let F be a θ-operator. Suppose either $F = P^*GP$ or $F = PGP^*$, where P is unitary, . Then, F is also a θ-operator.

Proof

By hypothesis, G is a θ-operator thus we have

$$G^*G(G^* + G) = (G^* + G)G^*G$$

That is

$$G^*GG^* + G^*G^2 = G^{*2}G + GG^*G \tag{1}$$

Also, we know that F and G are unitary hence we can write,

$$F = PGP^* \quad \text{which implies that} \quad F^* = PG^*P^*$$

Therefore,

$$F^*F = PG^*P^*PGP^* = PG^*GP^*$$

And

$$F^* + F = PG^*P^* + PTP^*.$$

Thus we have,

$$F^*F(F^*+F) = PG^*GP^*(PG^*P^*+PGP^*) = PG^*G^2P^*+PG^*G^2P^* \quad . \tag{2}$$

Similarly, the equation can be rewritten as;

$$(F^*+F)F^*F = (PG^*P^*+PGP^*)PG^*GP^* = PG^{*2}GP^*+PGG^*GP^* \quad . \tag{(3)}$$

From equation (1) we can note that;

$$PG^*GG^*P^*+PG^*G^2P^* = PG^{*2}GP^*+PGG^*GP^*.$$

While From (2) and (3) we have that.

$$F^*F(F^*+F) = (F^*+F)F^*F.$$

Hence, F is also a θ-operator.

Theorem 3.2.1.1.6: (Kiprop, 2016) Let $F,G \in B(H)$ such that G is a θ-operator and $F = UGU^*$ where U is an isometry. Then F is also a θ-operator.

Proof:

Since G is a θ-operator, we have:

$$G^*G(G^*+G) = (G^*+G)G^*G \implies G^*GG^*+G^*G^2 = G^{*2}G+GG^*G \quad (1)$$

Given $F = \mu G\mu^*$, we have $F^* = \mu G^*\mu^*$. Therefore,

$$F^*F = \mu G^*\mu^*\mu G\mu^* = \mu G^*G\mu^*$$

and

$$F^*+F = \mu G^*\mu^*+\mu G\mu^*.$$

Thus,

$$F^*F(F^*+F) = \mu G^*G\mu^*(\mu G^*\mu^* + \mu G\mu^*) = \mu G^*GG^*\mu^* + \mu G^*G^2\mu^* \quad (2)$$

Also,

$$(F^*+F)F^*F = (\mu G^*\mu^* + \mu G\mu^*)\mu G^*G\mu^* = \mu G^{*2}G\mu^* + \mu GG^*G\mu^* \quad (3)$$

Using equation (1), we get:

$$\mu G^*GG^*\mu^* + \mu G^*G^2\mu^* = \mu G^{*2}G\mu^* + \mu GG^*G\mu^*.$$

From equations (2) and (3), we have:

$$F^*F(F^*+F) = (F^*+F)F^*F.$$

Therefore, F is also a θ-operator.

Theorem 3.2.1.1.7 (Kiprop, 2016): Let $F, G \in B(H)$ such that G is a θ-operator and $F = U^*GU$ where U is a co-isometry. Then $F \in \theta$.

Proof:

Given G is a θ-operator,

$$G^*G(G^*+G) = (G^*+G)G^*G \implies G^*GG^* + G^*G^2 = G^{*2}G + GG^*G$$

Since $F = \mu^*G\mu$,

$$F^* = \mu^*G^*\mu$$

Therefore,

$$F^*F = \mu^*G^*\mu\mu^*G\mu = \mu^*G^*G\mu$$

and

$$F^* + F = \mu^*G^*\mu + \mu^*G\mu$$

Thus,

$$F^*F(F^*+F) = \mu^*G^*G\mu(\mu^*G^*\mu + \mu^*G\mu) = \mu^*G^*G(G^*+G)\mu$$

and

$$(F^*+F)F^*F = (\mu^*G^*\mu + \mu^*G\mu)\mu^*G^*\mu = \mu^*G^{*2}\mu + \mu^*GG^*\mu$$

Using the properties,

$$\mu^*G^*G(G^*+G)\mu + \mu^*G^*G^2\mu = \mu^*G^{*2}\mu + \mu^*GG^*\mu$$

From these results,

$$F^*F(F^*+F) = (F^*+F)F^*F$$

Thus, $F \in \theta$.

This study aims to extend the properties presented in theorems above to investigate if θ-operators preserves the properties of unitary quasi-equivalence. In particular, theorem 3.2.1.9 below will be extended to unitary quasi-equivalence of θ-operator.

Theorem 3.2.1.1.8: (Muteti, 2014). If $F \in B(H)$, then $G \in \theta$ if and only if F is almost similar to G for some normal operator G.

3.2.2 Determining properties of unitary quasi-equivalence on w-hyponormal operator

To establish the properties of unitary quasi-equivalence on w-hponormal operators, the characteristics of w-hyponormal operators will be explored. Furthermore, the definitions of hyponormal, Log-hyponormal, and w-hyponormal will play an important role in achieving this objective. The results on unitary equivalence and w-hyponormal will also be used in comparison to determine if unitary quasi-equivalence preserves the w-hyponormality of an operator. To show this the following theorems, definitions and properties will be pertinent to this investigation

Theorem 3.2.2.1: (Otieno 2007). an operator $G \in B(H)$ is said to be a W-hyponormal if it satisfy the condition $|\widetilde{G}| \geq |G| \geq |\widetilde{G}^*|$, where $|\widetilde{G}|$ denotes the Alugthe transformation.

Theorem 3.2.2.2: (Nzimbi & Luketero, 2020). Let F and GT be bounded linear operators defined on a Hilbert space H, then F and G are said to be unitary quasi-equivalent if there exists a unitary operator P such that:

$$F^*F = PG^*GP^*$$

$$FF^* = PGG^*P^*$$

Theorem 3.2.2.3: (Otieno 2007). If F and G^* are w-hyponormal operators with $\ker F \subset \ker F^*$ and $\ker G^* \subset \ker G$, and if there exists a quasiaffinity operator U such that $U \in \ker\delta F, G$, then F and G are unitarily equivalent normal operators.

Theorem 3.2.2.4: (Luketero & Khalagai, 2020). Let F be a hyponormal operator and G be another operator such that:

(i) $F = \mu G \mu^*$ where μ is an isometry, or

(ii) $F = \mu^* G \mu$ where μ is a co-isometry.

Proof:

$$Let F = \mu G \mu^* \tag{i}$$

By adjoint of an operator equation (i) can also be represented as following;

$$F^* = \mu^* G^* \mu. \tag{ii}$$

combining equation (i) and (ii), we have:

$$F^*F = (\mu^* G^* \mu)(\mu G \mu^*) = \mu^* G^* G \mu.$$

And

$$FF^* = (\mu G \mu^*)(\mu^* G^* \mu) = \mu G G^* \mu^*.$$

By the definition of normality, $F^*F \geq FF^*$ which further implies

$$\mu^* G^* G \mu \geq \mu G G^* \mu^*.$$

Pre-multiplying by μ and post-multiplying by μ^* gives:

$$G^*G \geq GG^*.$$

Hence, G is hyponormal.

similarly,

Let $F = \mu^*G\mu$.

This implies that $F^* = \mu^*G^*\mu$.

Further, this implies that

$$F^*F = (\mu^*G^*\mu)(\mu^*G\mu) = \mu^*G^*G\mu$$

And

$$FF^* = (\mu^*G\mu)(\mu^*G^*\mu) = \mu^*GG^*\mu.$$

From this, we note that

$$F^*F \geq FF^*$$

which implies

$$\mu^*G^*G\mu \geq \mu^*GG^*\mu. \tag{iii}$$

If equation iii is pre-multiplied by μ and post-multiplied by μ^*, it results in

$$G^*G \geq GG^*.$$

Hence, G is also hyponormal.

Also the underlying properties under polar decomposition properties will greatly assist in meeting this objective. This properties are as follow;

Theorem 3.2.2.5: (Mutuku, 2020). Let $G \geq 0$ and $F = U|F|$ be the polar decomposition of an operator F. For each $\alpha > 0$ and $\beta > 0$, the following statements hold:

(i) $U^*U(|F|^\beta G|F|^\beta)^\alpha = (|F|^\beta G|F|^\beta)^\alpha$

(ii) $U^*U(|F^*|^\beta G|F^*|^\beta)^\alpha = (|F^*|^\beta G|F^*|^\beta)^\alpha$

(iii) $(U|F|^{\beta}G|F|^{\beta}U^*)^{\alpha} = U(|F|^{\beta}G|F|^{\beta})^{\alpha}U^*$

(iv) $(U^*|F^*|^{\beta}G|F^*|^{\beta}U)^{\alpha} = U^*(|F^*|^{\beta}G|F^*|^{\beta})^{\alpha}U$

3.2.3 Establishing properties of unitary quasi-equivalence on (p,k)-quasi-hyponomal

To achieve this objective, properties of (p,k)-quasi-hyponormal operators will be useful. Some results on unitary equivalence and hyponormal operators will also play a significant role. In addition, the definitions and properties of p-hyponormal, k-hyponormal, normal, and (p,k)-hyponormal operators will be used. The following property will also play a great role in this study.

Theorem 3.2.3.1: (Kikete *et al.*, 2023). An operator $F \in B(H)$ is said to be (p,k)-quasihyponormal if $F^{*k}((F^*F)^p - (FF^*)^p)F^k \geq 0$, $0 < p \leq 1$, and a positive integer k.

Theorem 3.2.3.2: (Kikete *et al.*, 2023). Two bounded linear operators F and G in a Hilbert space H are considered to be (n,m)-unitarily quasiequivalent if there exists a unitary operator P such that $F^n \cdot F^m = P \cdot G^n \cdot G^m \cdot P^*$ and $F^m \cdot F^n = P \cdot G^m \cdot G^n \cdot P^*$.

Theorem 3.2.3.3: (Kikete *et al.*, 2023). Let F and $G \in B(H)$ be unitarily equivalent operators, and $F \in B(H)$ be a (n,m)-hyponormal operator such that:

(i) If $G = UFU^*$ where U is an isometry, then G is regarded to be (n,m)-hyponormal.

(ii) If $F = V^*GV$ where V is a co-isometry, then G is also (n,m)-hyponormal.

Proof.

(i) Let $T \in B(H)$ be such that $T = USU^*$ for U an isometry. Then

$$T^* = (USU^*)^* = US^*U^*$$

and

$$T^n = (USU^*)^n = (USU^* \cdot \ldots \cdot USU^*) \quad |\{z\} \text{ n - times} = US^nU^*$$

similarly,

$$T^m = US^mU^*$$

Therefore,

$$T^n(T^*)^m = US^nU^*(US^*U^*)^m = US^nU^*U(S^*)^mU^* = US^n(S^*)^mU^* \leq U(S^*)^mSU^* = U(S^*)^mU^*USU^* = (T^*)^mT^n$$

Hence, T is (n,m)- hyponormal.

(ii) Let $T \in B(H)$ be such that $T = V^*SV$ for V a co-isometry. Then

$$T^* = (V^*SV)^* = V^*S^*V$$

and

$$T^n = (V^*SV)^n = (V^*SV \cdot \ldots \cdot V^*SV) \quad |\{z\} = V^*S^nV$$

similarly,

$$T^m = V^*S^mV$$

Therefore,

$$T^n(T^*)^m = V^*S^nV(V^*S^*V)^m = V^*S^nVV^*(S^*)^mV = V^*S^n(S^*)^mV \leq V^*(S^*)^mSV = V^*(S^*)^mVV^*SV = (T^*)^mT^n$$

Hence, T is (n,m)- hyponormal.

In addition, some previous results on unitary equivalence, almost similarity of an operator and (n,m)-hyponormal operator will be used to compare if unitary quasi-equivalence preserves the (p,k)-quasi-hyponormality of an operator.

3.3 Ethical Consideration

In conducting research on the properties of unitary quasi-equivalence on selected operators, adherence to a set of ethical principles was essential in ensuring the integrity, reliability, and academic value of the research. This research adhered to the following research ethics.

Integrity and Accuracy: All findings and analyses was reported honestly, without fabrication, falsification, or inappropriate data manipulation. Efforts have been made to describe the methodologies and processes in detail, allowing for reproducibility and verification by other researchers.

Transparency and Openness: The research was transparent about the methods and criteria used for selecting operators for study. Any limitations of the research or potential biases discussed openly to provide a clear understanding of the context and scope of the findings.

Respect for Intellectual Property: Proper acknowledgment and citation of previous works has been ensured. This includes not only direct quotations but also the adaptation of ideas and concepts. The research has carefully avoided plagiarism and respect the copyright of others, giving credit where it is due.

NACOSTI Approval: The research permit was approved and obtained from the National Commission for Science, Technology, and Innovation (NACOSTI), ensuring full compliance with regulatory standards and transparency throughout the research process.

CHAPTER FOUR

UNITARY QUASI-EQUIVALENCE ON SOME SELECTED CLASSES OF OPERATORS IN HILBERT SPACES

4.1 Introduction

In this chapter, the outcome of the study objectives is presented. These objectives are divided into three sections. In Section 4.2, the properties of unitary quasi-equivalence on the θ-operator has been established. In Section 4.3, the properties of unitary quasi-equivalence on w-hyponormal operators has been presented, and finally, section 4.4 determines the properties of (p,k)-quasi-hyponormal operators.

4.2 Properties of Unitary Quasi-Equivalence and θ-Operators

In achieving the objective one the following class inclusion played an important role.

$$\text{Normal operators} \subseteq \theta\text{-operators}$$

An operator $F \in B(H)$ is normal operator if $G^*G = GG^*$. Nzimbi & Luketero (2021) noted that the unitary quasi-equivalence preserves the normality of an operator. Kiprop (2021) determined the properties of θ-operator on unitary equivalence operator, almost similar. The result showed that θ-operators preserves the properties of unitary equivalence and almost similarity of an operator. In addition, Kiprop (2021) determined that the class of Normal operator are also θ-operator. However the properties of unitary quasi equivalence on θ-operator has not been determined. Therefore, this study aimed to establish this gap.

The following information were useful in showing the results.

Lemma 4.2.1 (Nzimbi & Luketero, 2020). Let F and G be bounded linear operators defined on a Hilbert space H, then F and G are said to be unitary quasi-equivalent if there exists a unitary operator μ such that:

$$F^*F = \mu G^*G\mu^*$$

and

$$FF^* = \mu GG^* \mu^*$$

Lemma 4.2.2: (Muteti, 2014) If $F, G \in B(H)$ are such that $G \in \theta$ and F is almost similar to G, then $F \in \theta$.

Lemma 4.2.3: (Muteti, 2014). Let $F, G \in B(H)$ and let F be a θ-operator. Suppose either $F = \mu^* G\mu$ or $F = \mu G\mu^*$, where μ is unitary, . Then, G is also a θ-operator.

Lemma 4.2.4: (Nzimbi & Luketero, 2021). If Two operators F, G $\in B(H)$ are such that F is unitary quasi-equivalence to G and F is normal, then it implies that G is also a normal operator.

The following outcomes were established.

Theorem 4.2.5 Suppose that $F, G \in B(H)$ is such that F is unitary quasi-equivalence to G and $F \in \theta$. Then G is also a θ-operator.

Proof. From the hypothesis, F and G are unitarily quasi-equivalent, implying that

$$F^*F = \mu G^* G\mu^* \tag{4.1}$$

and

$$FF^* = \mu GG^* \mu^* \tag{4.2}$$

also F and G are projection and thus.

$$F^* = \mu G^* \mu^*$$

$$F = \mu G\mu^* \tag{4.3}$$

Also, we know that $F \in \theta$ and by Definition 1.5.31:

$$F^*F(F^* + F) = (F^* + F)F^*F \tag{4.4}$$

Substituting equations (4.1) and (4.2) into equation (4.4) yields equation (4.5):

$$\mu GG^* \mu^*(\mu G^* \mu^* + \mu G\mu^*) = (\mu G^* \mu^* + \mu G\mu^*)\mu GG^* \mu^* \tag{4.5}$$

Expanding the left and right sides of equation (4.5) results in:

$$\mu GG^*\mu^*\mu G^*\mu^* + \mu GG^*\mu^*\mu G\mu^* = \mu G^*\mu^*\mu GG^*\mu + \mu G\mu^*\mu GG^*\mu^* \tag{4.6}$$

However, since μ is unitary, then

$$\mu\mu^* = \mu^*\mu = I \tag{4.7}$$

Substituting (4.7) into (4.6) yields equation (4.8):

$$\mu GG^*IG^*\mu^* + \mu GG^*IG\mu^* = \mu G^*GIG^*\mu^* + \mu GIGG^*\mu^* \tag{4.8}$$

Pre-multiplying and post-multiplying equation (4.8) with μ^* on the left and μ on the right we get:

$$\mu^*\mu GG^*IG^*\mu^*\mu + \mu^*\mu GG^*IG\mu^*\mu = \mu^*\mu G^*GIG^*\mu^*\mu + \mu^*\mu GIGG^*\mu^*\mu$$

Thus,

$$IGG^*G^*I + IGG^*GI = IG^*GG^*I + IGGG^*I,$$

$$GG^*G^* + GG^*G = G^*GG^* + GGG^* \tag{4.9}$$

But GG^* is common on LHS and RHS of equation (4.9)

$$GG^*(G^* + G) = (G^* + G)GG^* \tag{4.10}$$

Equivalently;

$$FF^*(F + F^*) = (F + F^*)FF^* \tag{4.11}$$

Substituting equation (4.2) into equation (4.11) yields:

$$\mu GG^*\mu^*(\mu G\mu^* + \mu G^*\mu^*) = (\mu G\mu^* + \mu G^*\mu^*)\mu GG^*\mu^* \tag{4.12}$$

Expanding equation (4.12) by distributing $\mu GG^*\mu^*$ inside the brackets we get:

$$\mu GG^*\mu^*\mu G\mu^*\mu + \mu GG^*\mu^*\mu G^*\mu^* = \mu G\mu^*\mu GG^*\mu^* + \mu G^*\mu^*\mu GG^*\mu^*$$

But $\mu^*\mu = \mu\mu^* = I$; where I is the identity.

$$\mu GG^* I\mu^*\mu + \mu GG^* I\mu G^*\mu^* = \mu G\mu^*\mu GG^*\mu^* + \mu G^*\mu^*\mu GG^*\mu^* \tag{4.13}$$

Pre-multiplying and post-multiplying equation (4.13) with μ^* on the left and μ on the right we obtain equation (4.14) below;

$$\mu^*\mu GG^* I\mu^*\mu\mu + \mu^*\mu GG^* I\mu G^*\mu^*\mu = \mu^*\mu G\mu^*\mu GG^*\mu^*\mu + \mu^*\mu G^*\mu^*\mu GG^*\mu^*\mu$$

$$IGG^*IG^*I = IGG^*IG^*I + IGG^*IGG^*I$$

$$GG^*G + GG^*G^* = GGG^* + G^*GG^* \tag{4.14}$$

GG^* is common on LHS and RHS of the equation;

$$GG^*(G+G^*) = (G+G^*)GG^* \tag{4.15}$$

Equation (4.15) and (4.10) shows that G is a θ-Operator. $\square$

Example Consider an operator $F = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix}$ on the two-dimensional Hilbert space H_2. Demonstrate that F is a θ-operator.

Solution:

Given matrix F, the adjoint of $F = \begin{pmatrix} 6 & -4 \\ -4 & 2 \end{pmatrix}$. We need to show if $(F^*F)(F^*+F) = (F^*+F)(F^*F)$.

To verify this, we compute the following expressions:

$$F^*F = \begin{pmatrix} 6 & -4 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$$

$$F^*+F = \begin{pmatrix} 6 & -4 \\ -4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

$$(F^*F)(F^*+F) = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} -32 & 0 \\ 0 & -32 \end{pmatrix}$$

$$(F^*+F)(F^*F) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} -32 & 0 \\ 0 & -32 \end{pmatrix}$$

Thus, $(F^*F)(F^*+F) = (F^*+F)(F^*F)$.

Corollary 4.2.6: θ - operators are invariant under Unitary-quasi-equivalence.

4.3 Properties of Unitary Quasi-Equivalence and Hyponormal Operators

In achieving objective two the following class inclusion was considered:

Hyponormal $\subset$ Log-hyponormal $\subset$ w-Hyponormal $\subset$ Paranormal $\subset$ K-paranormal

Hyponormal $\subset$ p-Hyponormal $(0 < p < 1)$ $\subset$ w-Hyponormal $\subset$ Paranormal $\subset$ K-paranormal

(Nzimbi & Luketero, 2020) established that unitary quasi-equivalence preserves the hyponormality of an operator. In this study, the aim was to determine if unitary quasi-equivalence preserves the properties of w-hyponormality, log-hyponormality, and P-hyponormality. To determine this property, definitions 1.5.12, 1.5.13, and 1.5.16 were important. Additionally, the polar decomposition properties of an operator were essential. Using these properties, this study was able to establish the following results.

Theorem 4.3.1: Suppose that $F,G \in B(H)$ are unitary quasi-equivalent operators and F is a Log-Hyponormal operator, then G is also Log-Hyponormal.

Proof. Since F is unitary quasi-equivalent to G, then

$$F^*F = UG^*GU^* \tag{4.16}$$

$$FF^* = UGG^*U^* \tag{4.17}$$

Also, F is Log-Hyponormal implying that

$$\log(F^*F) \geq \log(FF^*) \tag{4.18}$$

Substituting equations (4.16) and (4.17) into (4.18) yields equation 4.19 as follows;

$$\log(UG^*GU^*) \geq \log(UGG^*U^*) \tag{4.19}$$

Pre-multiplying and post-multiplying both sides of (UGG^*U^*) and (UGG^*U^*) in equation (4.18) with U^* and U on the left and the right respectively yields;

$$\log(U^*UG^*GU^*U) \geq \log(U^*UGG^*U^*U) \tag{4.20}$$

But

$$U^*U = UU^* = I$$

Thus (4.20) becomes;

$$\log(IG^*GI) \geq \log(IGG^*I)$$

$$\log(G^*G) \geq \log(GG^*) \tag{4.21}$$

Equation (4.21) shows that G is the Log-Hyponormal operator. $\square$

Corollary 4.3.2: Unitary quasi-equivalence preserves the properties of log-hyponormal.

Theorem 4.3.3: Suppose $F, G \in B(H)$ are unitary quasi-equivalence operators if F is a p-hyponormal operator, then G is also P-Hyponormal operator.

Proof. Since F and G are unitary quasi-equivalent to each other, thus;

$$F^*F = UG^*GU^* \tag{4.22}$$

$$FF^* = UGG^*U^* \tag{4.23}$$

However, F is p-hyponormal thus by definition 1.5.20:

$$(F^*F)^P \geq (FF^*)^P \text{ for a positive number } P. \tag{4.24}$$

Replacing equation (4.22) and (4.23) into equation (4.24) results in:

$$(UG^*GU^*)^P \geq (UGG^*U^*)^P \tag{4.25}$$

By polar decomposition property;

$$G = U|G| \text{ where } U \text{ is a unitary operator.} \tag{4.26}$$

$$G^* = |G^*|U^* \tag{4.27}$$

Substituting equations (4.26) and (4.27) into equation (4.25) results in:

$$(U|G^*|U^*U|G|U^*)^P \geq (UU|G||G^*|U^*U^*)^P \tag{4.28}$$

From theorem 3.2.2.5, equation (4.28) yields equation (4.29):

$$U((|G^*|U^*U|G|)^P)G^* \geq U((U|G||G^*|U^*)^P)U^* \tag{4.29}$$

Post-multiplying and pre-multiplying both sides of equation (4.29) with U^* on the left and U on the right, we get (4.30) as follow:

$$U^*U((|G^*|U^*U|G|)^P)U^*U \geq U^*U((U|G||G^*|U^*)^P)U^*U \tag{4.30}$$

But $UU^* = U^*U = I$. Replacing the identity in equation (4.30):

$$I((|G^*|U^*U|G|)^P)I \geq I((U|G||G^*|U^*)^P)I$$

$$((|G^*|U^*U|G|)^P) \geq ((U|G||G^*|U^*)^P) \tag{4.31}$$

But from equation (4.27) and (4.28), (4.31) simplifies to:

$$(G^*G)^P \geq (GG^*)^P \tag{4.32}$$

And hence from (4.32) G is a p-hyponormal operator. □

Corollary 4.3.4: Unitary quasi-equivalence is p-hyponormal invariant.

Theorem 4.3.5: Let F and G be projection and unitary equivalent operators, if F is a w-hyponormal operator then G is also hyponormal.

Proof. From hypothesis F and G are unitary quasi equivalence, this implies that:

$$F^*F = UG^*GU^*$$

And

$$FF^* = UGG^*U^*$$

Also F is w-hyponormal which means that

$$|\widetilde{F}| \geq |F| \geq |\widetilde{F}^*| \qquad ((4.33))$$

Where;

$$F = U|F| \qquad ((4.34))$$

$$|\widetilde{F}| = |F|^{1/2}U|F|^{1/2} \qquad ((4.35))$$

$$|\widetilde{F}^*| = |F^*|^{1/2}U^*|F^*|^{1/2} \qquad ((4.36))$$

Substituting equation (4.34), (4.35) and (4.36) into equation (4.33) we get:

$$||F|^{1/2}U|F|^{1/2}| \geq |U|F|| \geq ||F^*|^{1/2}U^*|F^*|^{1/2}| \qquad ((4.37))$$

Since F is a projection,

$$F^* = UG^*U^*$$

And

$$F = UGU^*$$

Substituting F and F^* into equation (4.37) it yields to equation (4.38) as follow;

$$||UGU^*|^{1/2}U|UGU^*|^{1/2}| \geq |U|UGU^*|| \geq ||UG^*U^*|^{1/2}U^*|UG^*U^*|^{1/2}| \quad ((4.38))$$

From theorem 3.3.2, equation (4.38) can be rewritten as

$$|U(|G|^{1/2})U^*UU(|G|^{1/2})U^*| \geq |UU|G|U^*| \geq |U(|G^*|^{1/2})U^*U^*U(|G^*|^{1/2})U^*| \quad ((4.39))$$

since U is unitary then

$$U^*U = I$$

where I is identity and hence

$$U|(|G|^{1/2})U(|G|^{1/2})|U^* \geq UU||G||U^* \geq U|(|G^*|^{1/2})U^*I(|G^*|^{1/2})|U^* \quad ((4.40))$$

Pre multiplying both side of equation (4.40) by U^* on the left and post multiplying U on the right side we produce;

$$U^*U|(|G|^{1/2})U(|G|^{1/2})|U^*U \geq U^*UU||G||U^*U \geq U^*U|(|G^*|^{1/2})U^*I(|G^*|^{1/2})|U^*U$$

But

$$U^*U = I$$

$$I|(|G|^{1/2})U(|G|^{1/2})|I \geq IU||G||I \geq I|(|G^*|^{1/2})U^*I(|G^*|^{1/2})|I$$

Thus

$$|(|G|^{1/2})U(|G|^{1/2})| \geq U||G|| \geq |(|G^*|^{1/2})U^*(|G^*|^{1/2})| \quad ((4.41))$$

But

$$(|G|^{1/2})U(|G|^{1/2}) = |\widetilde{G}|$$

$$U|G| = T$$

And

$$|G^*|^{1/2})U^*(|G^*|^{1/2}) = |\widetilde{G}^*|$$

Thus equation (4.41) simplifies to

$$|\widetilde{G}| \geq |G| \geq |\widetilde{G}^*|$$

And hence G is also w-hyponormal. □

Corollary 4.3.6: w-hyponormal is unitary quasi-equivalence invariant.

4.4 Properties of Unitary Quasi-Equivalence and (p,k)-quasi-Hyponormal Operators

To prove objective three, the following containment was pertinent in this study.

i. self-adjoint ⊂ normal ⊂ hyponormal ⊂ p-hyponormal ⊂ p-quasihyponormal ⊂ (p,k)-quasihyponormal

ii. hyponormal ⊂ quasihyponormal ⊂ k-quasihyponormal ⊂ (p,k)-quasihyponormal

iii. hyponormal ⊂ quasihyponormal ⊂ k-quasihyponormal ⊂ (p,k)-quasihyponormal

Kikete *et al*., (2020), worked on (n,m)-hyponormal operator and established that the operator preserves the unitary quasi-equivalence. Additionally, (n,m)-hyponormal operator was shown to preserve the (n, m)-unitary quasi-equivalence. However this study aimed to determine the property of p-quasi-hyponormal, k-hyponormal and (p,k)-quasi-hyponormal on unitary-quasi -equivalence.
The concept of polar decomposition became important in showing results and in particular, in dissociating the powers. definition 1.5.20, 1.5.21, 1.5.15 and 1.5.22 also played pivotal role in establishing the results. Through these efforts the following results were established.
Theorem 4.4.1 Let $F,G \in B(H)$ be projection and unitary quasi-equivalence. If F is a p-quasi-hyponormal operator, then G is also p-quasi-hyponormal.

Proof. Since F is unitary quasi-equivalent to G, then:

$$F^*F = UG^*GU^* \tag{4.42}$$

and

$$FF^* = UGG^*U^* \tag{4.43}$$

Also, F and G are projections:

$$F = UGU^*$$

$$F^* = UG^*U^*$$

F is p-quasi-hyponormal, and by Definition 1.5.21:

$$F^*((F^*F)^p - (FF^*)^p)F \geq 0 \text{ for } 0 < p < 1 \tag{4.44}$$

Substituting equations (4.42) and (4.43) into equation (4.44), gives:

$$UG^*U^*((UG^*GU^*)^p - (UGG^*U^*)^p)UGU^* \geq 0 \tag{4.45}$$

Substituting (4.26) and (4.27) into equation (4.45) establishes the following;

$$U|G^*|U^*U^*((U|G^*|U^*U|G|U^*)^p - (UU|G||G^*|U^*U^*)^p)UU|G|U^*U^* \geq 0 \tag{4.46}$$

and on applying Theorem 3.2.2.5, the equation reduces to:

$$U|G^*|U^*U^*(U(|G^*|U^*U|G|)^pU^* - U(U|G||G^*|U^*)^pU^*)UU|G|U^*U^* \geq 0 \tag{4.47}$$

Pre-multiplying on both side of equation (4.47) from left with U^* and post-multiplying from the right with U;

$$U^*U|G^*|U^*U^*U(U^*U(|G^*|U^*U|G|)^pU^*U - U^*U(U|G||G^*|U^*)^pU^*U)U^*UU|G|U^*U^*U \geq 0$$

But

$$U^*U = I$$

$$I|G^*|U^*I(I(|G^*|U^*U|G|)^pI - I(U|G||G^*|U^*)^pI)IU|G|U^*I \geq 0$$

$$|G^*|U^*((|G^*|U^*U|G|)^p - (U|G||G^*|U^*)^p)U|G|U^* \geq 0 \tag{4.48}$$

But:

$$U|G| = G \tag{4.49}$$

and

$$|G^*|U^* = G^* \tag{4.50}$$

Substituting (4.49) and (4.50) back into equation (4.48), the equation reduces to:

$$G^*((G^*G)^p - (GG^*)^p)G$$

And thus from definition 1.5.21, G is also p-quasi-hyponormal. □

Theorem 4.4.2 Let $F, G \in B(H)$ be projection and unitary quasi-equivalence. If F is a k-quasi-hyponormal operator, then G is also k-quasi-hyponormal.

Proof. F and G are unitary quasi-equivalence thus;

$$F^*F = UG^*GU^* \tag{4.51}$$

$$FF^* = UGG^*U^* \tag{4.52}$$

Also, F and G are projections and thus

$$F = UGU^* \tag{4.53}$$

$$F^* = UG^*U^* \tag{4.54}$$

But F is k-quasi-hyponormal thus by definition 1.5.22

$$F^{*k}((F^*F) - (FF^*))F^k \geq 0 \text{ for a positive number } k \text{ i.e., } k > 0 \tag{4.55}$$

Substituting equations (4.51) and (4.52) and expanding;

$$(UGU^*)^k((UG^*GU^*) - (UGG^*U^*))(UGU^*)^k \geq 0 \tag{4.56}$$

Substituting equation (4.26) and (4.27) into equation (4.56)

$$(U|G^{*}|U^{*}U^{*})^{k}UG^{*}GU^{*}(UU|G|U^{*})^{k} \geq (U|G^{*}|U^{*}U^{*})^{k}(UGG^{*}U^{*})(UU|G|U^{*})^{k} \tag{4.57}$$

Applying Theorem 3.2.2.5 in equation (4.57)

$$U(|G^{*}|U^{*})^{k}U^{*}UG^{*}GU^{*}U(U|G|)^{k}U^{*} \geq U(|G^{*}|U^{*})^{k}U^{*}UGG^{*}U^{*}U(U|G|)^{k}U^{*} \tag{4.58}$$

Pre-multiplying equation (4.58) from left with U^{*} and post-multiplying it with U from the right

$$U^{*}U(|G^{*}|U^{*})^{k}U^{*}UG^{*}GU^{*}U(U|G|)^{k}U^{*}U \geq U^{*}U(|G^{*}|U^{*})^{k}U^{*}UGG^{*}U^{*}U(U|G|)^{k}U^{*}U \tag{4.59}$$

But $U^{*}U = I$

$$I(|G^{*}|U^{*})^{k}IG^{*}GI(U|G|)^{k}I \geq I(|G^{*}|U^{*})^{k}IGG^{*}I(U|G|)^{k}I \tag{4.60}$$

Replacing $|G^{*}|U^{*}$ with G^{*} and $U|G|$ with G equation (4.60) becomes

$$(G^{*})^{k}(G^{*}G)(G)^{k} \geq (G^{*})^{k}(GG^{*})(G)^{k} \tag{4.61}$$

Equation (4.61) can be simplified to

$$G^{*k}((G^{*}G - GG^{*})G^{k} \geq 0 \tag{4.62}$$

And thus by definition 1.5.22 G is also k-quasi-hyponormal. □

Theorem 4.4.3 Let $F,G \in B(H)$ be projection and unitary quasi-equivalence operators. If F is a (p,k)-quasi-hyponormal operator, then G is also a (p,k)-quasi-hyponormal operator.

Proof. From hypothesis, F and G are projection and unitary quasi-equivalence operators, thus from definition 1.5.32:

$$F^{*}F = UG^{*}GU^{*} \tag{4.63}$$

and

$$FF^{*} = UGG^{*}U^{*} \tag{4.64}$$

$$F = UGU^* \tag{4.65}$$

$$F^* = UG^*U^* \tag{4.66}$$

Since F is (p,k)-quasi-hyponormal, then by definition 1.5.23:

$$F^k((F^*F)^p - (FF^*)^p)F^k \geq 0, \quad 0 < p \leq 1 \text{ and a positive integer } k. \tag{4.67}$$

Substituting equations (4.63), (4.64), (4.65), and (4.66) into (4.67)we yield equation (4.68) as follows;

$$(UG^*U^*)^k((UG^*GU^*)^p - (UGG^*U^*)^p)(UGU^*)^k \geq 0 \tag{4.68}$$

By polar decomposition property, equation (4.68) becomes:

$$(U|G^*|U^*U^*)^k\,((U|G^*|U^*U|G|U^*)^p - (UU|G||G^*|U^*U^*)^p)\,(UU|G|U^*)^k \geq 0$$

By theorem 3.2.2.5 we have:

$$U(|G^*|U^*)^kU^*\,(U(|G^*|U^*U|G|)^pU^* - U(U|G||G^*|U^*)^pU^*)\,U(U|G|)^kU^* \geq 0 \tag{4.69}$$

Pre-multiplying the left side of equation (4.69) by U^* and post-multiplying by U on the right:

$$U^*U(|G^*|U^*)^kU^*U\,(U^*U(|G^*|U^*U|G|)^pU^*U - U^*U(U|G||G^*|U^*)^pU^*U)\,U^*U(U|G|)^kU^*U \geq 0$$

But $U^*U = I$, implying:

$$I(|G^*|U^*)^kI\,(I(|G^*|U^*U|G|)^pI - I(U|G||G^*|U^*)^pI)\,I(U|G|)^kI \geq 0$$

and further Implies that:

$$(|G^*|U^*)^k\,((|G^*|U^*U|G|)^p - (U|G||G^*|U^*)^p)\,(U|G|)^k \geq 0 \tag{4.70}$$

Replacing $|G^*|U^*$ with G^* and $U|G|$ with G, equation (4.70) becomes:

$$G^k\left((G^*G)^p-(GG^*)^p\right)G^k \geq 0$$

Hence, G is also a (p,k)-quasi-hyponormal operator. $\square$

Corollary 4.4.4: The class of (p,k)-quasi-hyponormal is invariant under unitary quasi-equivalence of operator.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

Near equivalence, Metric equivalence, quasi-similarity, unitary equivalence, and unitary-quasi-equivalence, among others, are examples of equivalence relations of operators in Hilbert spaces. In the exploration of these operators by different scholars, they have been shown to preserve or not preserve the properties of underlying operators and under which conditions do they preserve. Unitary-quasi-equivalence, an example of an equivalent relation operator, was pioneered by Othman (1996) under the concept of near equivalence. It has seen some exploration since then. For instance, Nzimbi & Khalagai (2020) determined that it preserves normality, binormality, unitary, and hyponormality.

This study aimed to study unitary quasi-equivalence on θ-operator, w-hyponormality, and (p,k)-quasi-hyponormality and their classes. In the study of properties related to unitary quasi-equivalence and θ-operators, the class inclusion became pertinent in understanding the properties of θ-operators. Through rigorous systematic review, the study was able to establish that unitary-quasi-equivalence preserves the properties of θ-operators as shown in Theorem 4.2.1.

Nzimbi & Khalagai (2020) established that unitary-equivalent operator preserves the hyponormality of an operator. Kikete *et al.* (2023) worked on (n,m)-hyponormal operator and established that it preserves the properties of unitary-quasi-equivalence. However, this study established that unitary quasi-equivalence preserves log-hyponormality, p-hyponormality, and w-hyponormality of an operator. This means if two operators are unitary quasi-equivalent and one of them is either log-hyponormality, p-hyponormality, or w-hyponormality then so is the other. On another class of hyponormal operator, with the class inclusion

$$\text{hyponormal operators} \subseteq \text{p-quasi-hyponormal operators} \subseteq (p,k)\text{-quasi-hyponormal operators}$$

$$\text{hyponormal operators} \subseteq \text{k-quasi-hyponormal operators} \subseteq (p,k)\text{-quasi-hyponormal operators}$$

The following results were established: If two bounded and linear operators are unitary-quasi-equivalence and one of them is p-quasi-hyponormal, k-quasi-hyponormal, and (p,k)-quasi-hyponormal, then so is the other. This means they preserve equivalence as shown in Theorem 4.4.1, 4.4.2, and 4.4.3.

5.2 Recommendation

This study aimed to determine the properties of unitary quasi-equivalence on θ-operators, the class of w-hyponormality, p, k, and (p,k)-quasi-hyponormal operators. However, the properties of these operators can be extended to other equivalent relation operators. For instance, the properties of the three operators have not been established for unitary-equivalence, metric equivalence, and quasi-similarity. Additionally, the findings of this study can be extended to other operators such as (p,k)-quasi-posinormal operators. Thus to enhance the knowledge of this subject, it is recommendend that these areas be looked at.

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